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RAY CALCULATIONS OF OCEAN SOUND CHANNELS USING A POCKET PROGRAM--ETC(U)

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RAY CALCULATIONS OF OCEAN SOUND CHANNELS USING A POCKET PROGRAMMABLE CALCULATOR AND EXTENDED FORMS OF THE HIRSCH-CARTER MATHEMATICAL MODEL WITH TABLES OF THE INCOMPLETE BETA FUNCTION.

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Supplement follows main report in this volume

ABSTRACT

Formulas for curve fitting and ray computation using compound models made up of several different layers of form $C^2 = Co^2 \left(1 - |\alpha Z|^{\beta}\right)^{-1}$ are presented. Examples of computation by pocket programmable calculator on two Sargasso Sea profiles, one from the center of a cold ring eddy are given. Necessary tables of the incomplete beta-function and calculator programs are included in a supplement.

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L. Baxter, II

I. Introduction

Hirsch and Carter[1] have given closed form expressions for range and travel time of integral numbers of cycles of ray paths in the family of symmetrical profiles given by:

$$c^2 = c_0^2 \left(1 - |\alpha Z|^\beta\right)^{-1} \tag{1}$$

where c is the speed of sound at the vertical distance Z from the depth at which the speed is c_0 and α and β are parameters. Pedersen and Gordon[2], Weinberg[3], Stewart[4], and others have developed the concept of fitting realistic acoustic profiles with layers of various curved profile segments while matching the speed and slope of the speed at the layer interfaces. This technique prevents the calculation of "false caustics" and other artifacts associated with less sophisticated profile fits and minimizes the number of layers needed to represent a natural sound speed profile realistically.

Equation 1 can be used with different parameters in each layer of a multilayer profile fit. The geometry of a ray in a layer may be understood by referring to Figure 1 in which a ray from the reference level ($c = c_0$) is refracted as the sound propagates through higher speed levels, and Z (always positive) is the absolute value of the depth difference from the reference level. With Z defined in this way and α always positive, Equation 1 may be rewritten as:

$$c^2 = c_0^2 \left(1 - (\alpha Z)^\beta\right)^{-1} \tag{1A}$$

The closed form expressions given by Hirsch and Carter[1] for range and travel time apply only to integral multiples of rays from the reference level

to the vertex, z_v , where the ray becomes horizontal. The portions of this path which we need to compute for rays that traverse several layers can however be expressed almost as simply in terms of incomplete beta-functions which have been tabulated[5]. Convergent series for computing them directly have also been published[6].

For profile models consisting of no more than four layers of this type with no more than two layers on each side of the sound channel axis, it is not too difficult or tiresome to do ray computations with a medium capacity pocket programmable calculator and tables. I have done such calculations, fitting various natural asymmetric profiles with approximations consisting of two or three layers, using different parameters of Equation 1 in each layer and matching speed of sound and its derivative at the interfaces between layers. In this paper I outline the methods and give sample results for two profiles from the Sargasso Sea, one from the center of a cold eddy and one outside of any eddies. I also outline an approximate method for calculating rays that propagate through a small horizontal gradient of sound speed and a method of calculating range annoted ray angle diagrams.

II. Incomplete Beta-function and Calculator Programs for Acoustic Ray Computations

Although an extensive table[6] of the incomplete beta-function is available, the most important range of the variables for our purpose is too sparsely covered. A supplement* to the present paper tabulates the necessary detail. The supplement also contains a Fortran program for generating any other values that may be required, and operating instructions and listings of the curve fitting and ray computation programs for the Texas Instruments SR56 calculator which I used. The calculator programs could be applied with little change to any equivalent or larger calculator

using algebraic operating system.

III. The Geometry of Sound Speed Profile Layers in Which $c^2 = c_o^2 (1-(\alpha Z)^\beta)^{-1}$

We need the slope dc/dz in order to match different layers at the interfaces. Differentiating Equation 1A, we have:

$$\frac{dc}{dz} = \frac{\alpha \beta c_o (\alpha Z)^{\beta-1}}{2(1-(\alpha Z)^\beta)^{3/2}} \tag{2}$$

As we shall show later, the ray computations are simpler if we can fit the profile with layers in which the minimum speed of sound is equal to C_o and occurs at one interface. Therefore the limit of the slope, dc/dz, as Z approaches zero is an important parameter. Remembering that $\alpha > 0$ and $Z \geq 0$ we have three cases:

Case 1. If $\beta > 1$, $dc/dz \rightarrow 0$ as $Z \rightarrow 0$, regardless of the values of C_c and α.

Case 2. If $\beta = 1$, $dc/dz \rightarrow \alpha c_o/2$ as $Z \rightarrow 0$.

Case 3. If $\beta < 1$, $dc/dz \rightarrow \infty$ as $Z \rightarrow 0$ regardless of the values of C_o and α.

For realistic sound speed profiles of ocean sound channels, Case 1 layers should be used to interface at the axis or minimum of the sound speed profile; the outer layers may belong to Case 2 or Case 3. This statement will be clarified by the following more detailed discussion of layer geometry for realistic values of α, β, C_o and Z.

For refracted rays, the sound speed does not usually exceed about 102% of the axial speed of about 1.493 km/sec. The shape of the curves of Equations 1 and 2 in the range of the parameters for a 2% change in sound speed is most critically dependent on β, and reasonable changes in β may call for changes in α over a range of 10^{28} while changes in units

of Z, or depth variations of actual profiles, change α by much smaller ratios. To show the shape changes due to β on the same axes for various values (Figures 2 and 3), I use arbitrary units for Z with α adjusted to produce a maximum sound speed change of about 2% at Z=1. With these conventions, the order of magnitude of α is approximately that which would be realistic for Z kilometers.

Figure 2 shows the geometry of Equation 1 while Figure 3 shows that of Equation 2, i.e. the slope for the same values of the parameters. In these figures curves 1-6 belong to Case 3, curve 7 is Case 2 and curves 8-11 are Case 1. For Case 1 and Case 2 dc/dz increases with increasing Z, but in Case 2 the increase is not significant within the 2% change in sound speed. Case 2 approximates to a straight line and is the only case for which dc/dz at c_o is controlled by the parameter α . Case 3 layers are the only ones in which dc/dz decreases with increasing Z. They are somewhat more difficult to use because matching dc/dz to a lower velocity adjoining layer requires an interface at $Z>0$. The process will be explained in the next section of this report.

IV. Fitting Ocean Sound Speed Profiles Using Hirsch-Carter Type Layers

We can now see that the conditions of Pedersen and Gordon[2], matching sound speed and slope, are met by asymmetrical profiles (see Figures 4 and 5) consisting of a Case 1 Hirsch-Carter type upper layer (designated by U) meeting a Case 1 lower layer (designated by L) at the sound channel axis. If the designations are used as subscripts and the subscript A refers to the axis of the sound channel $(c_o)_U = (c_o)_L = c_A$, but $\alpha_U \neq \alpha_L$ and $\beta_U \neq \beta_L$. The U and L layers must belong to Case 1 because only a zero value of dc/dz at $c_o = c_A$, or minimum sound velocity, can give a common tangent at Z=0.

Figures 4 and 5 illustrate sound speed profiles from the Sargasso Sea. The profile in Figure 4 is from the center of a cold ring eddy; that in Figure 5 is from a location undisturbed by the eddy. To fit each of these with a U and L layer I used a calculator program which iterates from a trial value of β to place a Hirsch-Carter type curve through C_0 at the axis and points 1 (C_1 , Z_1) and 2 (C_2 , Z_2) each marked with an X. The dots in Figures 4 and 5 indicate the resulting fit.

Where the fit is not perfect the exact values of α and β depend of course on the chosen points 1 and 2, and an equally good or better fit might appear from a different choice. It simplifies the calculations if β corresponds exactly to a tabulated value in the supplement or in Pearson[5]. Therefore it is worthwhile to try such a value, as close as possible to the calculated β . α is then recalculated to fit a point near the middle of the layer. The fit of the new values of α and β is checked over the measured profile. The values are adopted if the fit seems good enough.

The fit is improved if one does not try to cover too great a range of depths with one layer. The depth range can be subdivided into additional layers but meeting the conditions of equality of sound speed and its derivative are somewhat more complicated when the interface is not at the sound channel axis. The sound ray calculations also become more complicated because the rays must be divided into segments that traverse the various layers. Inspecting Figures 4 and 5 we see that the fit above the axis would be much improved by another layer. In Figures 6 and 7 an "M" layer above the U layer has been added to each of these profiles.

Both in the curve fitting and in the calculations, to be discussed later it will be useful to think of a separate space for each layer.

In Figure 1 the layer interfaces with adjoining layers may occur at Z_Q and Z_S, but the layer space and its coordinate system extends beyond the portion $Z_Q - Z_S$ that actually fits approximately to the real profile. The ray segments OZ_Q and $Z_S Z_V$ in the layer space are only auxilliary constructs for computing the segment $Z_Q Z_S$ which corresponds to the real ray in the layer. In the following discussion the subscripts U, M, or L are intended to indicate the space in which a coordinate is measured.

Figure 6 is an example of a profile that can be fitted with $\beta_M = 1$. $(c_0)_M$ is set equal to C_U at the chosen interface. $(dc/dz)_U$ at the interface is calculated from Equation 2. Then:

$$\alpha_M = 2\,(dc/dz)_U \,/\, (c_0)_M \tag{3}$$

In Figure 7 $\beta_M < 1$. Since $(dc/dz)_M \to \infty$ as $Z_M \to 0$, slopes can be matched only if the reference velocity, $(c_0)_M$, is less than the velocity C_U at the interface and $Z_M > 0$ at the same place.

The fit was carried out as follows: The layer interface in U was chosen near a point of inflection of the empirical profile. C_U and $(dc/dz)_U$ at the interface were calculated. With some trial and error a layer thickness (Z_M) max, and layer parameters β_M, α_M and $(c_0)_M$ were selected to approximate the curvature and maximum sound speed in the empirical profile. (Z_M) interface was then computed from $(dc/dz)_u = (dc/dz)_M$ interface using a program based on Equation 2. The program iterates from a trial value Z_t to a more accurate value of (Z_M) interface.

V. Solutions for General Ray Segments in the Hirsch-Carter Model

For our purposes it is useful to put the equations of Hirsch and Carter[1] in slightly different form. Within a layer described by Equation 1, a ray is designated by the angle θ_0 at velocity c_0 (see Figure 1). It vertexes at $Z = Z_V$ where

$$Z_V = (\sin\theta_0)^{2/\beta} / \alpha \tag{4}$$

The variable φ defined by Hirsch and Carter[1] as

$$\varphi = (\alpha Z)^{\beta} / \sin^2\theta_0 \tag{5}$$

may also be expressed at Z by

$$\varphi = \left(Z / Z_V \right)^{\beta} \tag{6}$$

The range, R_{0Z}, covered by the ray segment from 0 to Z, can be written:

$$\begin{aligned} R_{0Z} &= \frac{Z_V}{\beta \tan\theta_0} \int_0^{\varphi} (1-x)^{\frac{1}{2}} x^{\frac{1}{\beta}-1} \, dx \\ &= \frac{Z_V}{\beta \tan\theta_0} \cdot B\left(\frac{1}{\beta}, \frac{1}{2}\right) \cdot I_{\varphi}\left(\frac{1}{\beta}, \frac{1}{2}\right) \end{aligned} \tag{7}$$

where B is the complete beta function and I is the ratio of the incomplete to complete beta function. Let

$$B_1 = B\left(\frac{1}{\beta}, \frac{1}{2}\right) \quad \text{and} \quad I_1 = I_{\varphi}\left(\frac{1}{\beta}, \frac{1}{2}\right)$$

then

$$R_{0Z} = Z_V \cdot B_1 \cdot I_1 / \beta \tan\theta_0 \tag{8}$$

The range, R_{ZZ_v}, can be written

$$R_{ZZ_v} = Z_v \cdot B(1-I_1)/\beta \tan\theta_o \qquad (9)$$

Let $B_2 = B\left(1+\frac{1}{\beta}, \frac{1}{2}\right)$ and $I_2 = I_y\left(1+\frac{1}{\beta}, \frac{1}{2}\right)$.

The travel times that correspond to the ranges of Equations 8 and 9 may be written

$$T_{oZ} = \frac{R_{oZ}}{C_o \cos\theta_o}\left\{1 - \sin^2\theta_o \frac{I_2 B_2}{I_1 B_1}\right\} \qquad (10)$$

and

$$T_{ZZ_v} = \frac{R_{ZZ_v}}{C_o \cos\theta_o}\left\{1 - \sin^2\theta_o \frac{(1-I_2) B_2}{(1-I_1) B_1}\right\} \qquad (11)$$

The values of the complete beta function, B_1 and B_2 are constants for a given layer of a profile. They may be calculated or taken at once from the tables. The relative values, I_1 and I_2, of the incomplete beta function depend on θ_o and Z through Equations 4 and 6, and the tables. The range and travel time of any segment such as Q-S (Figure 1) is easily obtained as a difference between values computed by the above equations.

Programs for Equations 4, 6, 8, 9, 10 and 11 fit easily in the SR56 calculator when I_1 and I_2 are entered from tables. Note that if $Z = Z_v$, I_1 and $I_2=1$ and Equations 8 and 10 reduce to those given by Hirsch and Carter[1]. To obtain total ranges for N axis crossings the values can of course be multiplied by 2N as is done by Hirsch and Carter.

VI. Calculation vs Axial Angle of Range and Travel Time at the End of Loops Above and Below the Sound Channel Axis and at the End of a Complete Cycle

Since the classical ray acoustics paper of Ewing and Worzell[7], sound channel computations have often been presented by plots of range and travel time of loops above and below the axis and of a full ray cycle, all vs axial angle as the independent variable. These data are presented for the three-layer fits to the eddy and Sargasso Sea profiles in Figures 8, 9, 10 and 11. The procedure for calculating these plots is described: first for the simpler case of Figures 8 and 9 where $\beta_M = 1$, and then the modifications for Figures 10 and 11 where $\beta_M < 1$.

Axis to axis loops that do not penetrate into a second layer are computed by straight-forward application of Equations 4, 8 and 10 with the factor 2N equal to 2, I_1 and I_2 equal to 1, and θ_0 equal to the axial angle, θ_A. On the lower side of the axis where the profile fit has no second layer, the full range of axial angles may be covered this simply.

The Z coordinate of the interface in U layer space can be written $(z_i)_U$. When $(z_v)_U$ becomes greater than $(z_i)_U$, the calculations can be simplified if θ_A is chosen so that γ is exactly equal to a value of X printed in the tables of the incomplete beta-function. Omitting the subscript U:

$$z_v = z_i / X^{1/\beta} \tag{12}$$

The axial angle, θ_A, for this ray is given by:

$$\theta_A = \arcsin\left[(\alpha z_v)^{\beta/2}\right] \tag{13}$$

When $\beta = 1$, the reference sound speed, $(c_0)_M$, for the M layer is

equal to $(c_i)_U$, the sound speed at the interface. The reference angle in the M layer is calculated by

$$(\theta_o)_M = \arccos\left[\frac{(c_o)_M}{c_A}\cos\theta_A\right] \tag{14}$$

With θ_A and $(\theta_o)_M$ tabulated, one returns to the program for Equations 4, 8 and 10. Taking I_1 and I_2 directly without interpolation from the tables, range and travel time for the portion of the loop in the U layer is computed using the θ_A just found. I_1 and I_2 in the layer equal 1 in this case. The portion of the ray in the M layer is computed using the $(\theta_o)_M$ equivalent to θ_A from Equation 14. The values of range, travel time, and distance, Z_V , in the U and M layers are added to obtain the values plotted for the ray loop above the axis. These range and travel time values are added to those computed for the same θ_A below the axis to obtain the values for the full ray cycle.

When $\beta_M < 1$, as in Figures 10 and 11, the segments in M must be computed differently. $(c_o)_M \neq (c_i)_U$ and Equations 9 and 11 must be used instead of 8 and 10. I_1 and I_2 in the M layer are not equal to 1. One could use directly the $(\theta_o)_M$ that correspond to θ_A by Equation 14, but one would have to interpolate in the tables for I_1 and I_2.

It is easier to defer the interpolation, doing it in $(\theta_o)_M$ at a later stage. This is done by using Equations 12 and 13 on the M layer after they have been used on the U layer. Equation 13 however is understood as:

$$(\theta_o)_{Mx} = \arcsin\left[(\alpha Z_V)^{\beta/2}\right]_M \tag{15}$$

where $(\theta_o)_{Mx}$ is the value of $(\theta_o)_M$ that corresponds to a tabulated value of X and not to θ_A. Range, travel time, and Z_V computed in the M layer for $(\theta_o)_{Mx}$ are interpolated to find the values for $(\theta_o)_M$ that do correspond to θ_A. The results are added to those for the U layer as before.

VII. Calculation of Arrival Times for the Eigen Rays for a Source and Receiver

This problem is merely an extension of the techniques used in Section VI. First one adds the appropriate segments to obtain a plot of range vs axial angle for the source and receiver depths and the possible types of path. Figure 12 is an example of this step. One interpolates to find the axial angles of each path at a given range. Then range and travel time are computed for these axial angles. Due to limited precision in the first interpolation the ranges will differ slightly but the average sound speeds will be correct for each path at the desired range. A second linear interpolation will adjust all the travel times to the correct range. Table I illustrates the result at a range of 705 km in Figure 12 and rays of order 14 through 16.

TABLE I

Travel Time at 705 km of rays of order 14, 15 and 16 in Sargasso Sea Profile

	T	ΔT	N
7.16964	471.65950	0.0000000	16
7.21127	471.65254	0.0069642	16
7.40994	471.604778	0.054726	16
7.46963	471.59383	0.1001213	16
8.29557	471.438324	0.2211799	15
8.40314	471.414127	0.2453772	15
8.57449	471.36485	0.2946512	15
8.69705	471.33525	0.324255	15
9.70808	471.11683	0.54267	14
9.87478	471.068211	0.59129	14
10.03846	471.015474	0.64403	14
10.21703	470.959929	0.69958	14

VIII. Calculation of the Relative Intensity or Focusing Factor

a. Relative Intensity Except at Caustics

Brekhovskikh[8] defines a "focusing factor" $f=I/I_o$, the ratio of the acoustic intensity I at a given point in the homogeneous medium to the acoustic intensity I_o in a homogeneous medium at the same distance. He shows that when $R \gg Z$ and the point is not a caustic.

$$f = R / \sin\theta_P \, (dR/d\theta_A)_P \tag{16}$$

where θ_P is the horizontal angle at the given point and the derivative is evaluated for the ray that passes through the point.

$$\theta_P = \arccos\left(c\,(\cos\theta_A)/c_o\right) \tag{17}$$

$(dR/d\theta_A)_P$ may be obtained graphically as the slope from a plot like Figure 12.

b. Relative Intensity at a Caustic

In Figure 12 the four rays of a given order appear in two pairs. Each pair appears to join at a point for an axial angle slightly less than 7°. The scale is too coarse to show the detail in the neighborhood of the supposed point which is really the location of a caustic. Figure 13 shows the "point" of the lower pair of order 17 on a greatly expanded scale. The method of calculating this detail will be discussed after I outline its application.

We have been interested in comparing the relative intensity of caustics in differing profiles but at a given range. Although ordinary ray theory fails at a caustic, Brekhovskikh[9] discusses a method of calculating intensity at a caustic from ray parameters. The full expression involves an Airy function and is rather complicated, but to compare the maxima of caustics under different conditions without computing the true relative

intensity at any point the expressions can be shortened. In the notation of this paper, and discarding factors that don't vary much in actual sound velocity profiles, relative intensity at a given large range and a given acoustic frequency is inversely proportional to tan θ_A sin $\theta_P \left(d^2R/d\theta_A^2\right)_P^{2/3}$ where θ_P (see Equation 17) is the angle with the horizontal of a ray tangent to the caustic. The method of computing the data for Figure 13 enables us to evaluate the derivative $\left(d^2R/d\theta_A^2\right)_P$; the other factors are obvious.

To calculate the range of a ray near the caustic, we measure, in Figure 10, the slope, S_f, and intercept I_F, of the full cycle ray that vertexes at the receiver depth. The values are: S_F=3 km/degree, I_f=22 km. We measure also the slope, S_u, and intercept I_u of the upper branch. S_u=0 km/degree. I_u=10 km. We then calculate range from the vertex vs axial angle for segments to the receiver depth of rays that vertex slightly shallower. The result appears as Figure 14. We note that the range increment, r_x, due to this segment is approximately the parabola

$$r_x^2 = K\left(\theta_A - (\theta_A)_P\right) \tag{18}$$

where, in the given example, K=2.9781 and $(\theta_A)_P$, the axial angle of the ray that vertexes at the receiver depth, =6.892 degrees.

Let the angular difference, $\left[\theta_A - (\theta_A)_P\right] = \varphi$. Total range of a ray of order N in the vicinity of the vertex can be written as follows:

$$R = Q\left(I_u + S_u \cdot \varphi\right) + N\left(I_f + S_f \cdot \varphi\right) \pm K^{\frac{1}{2}} \varphi^{\frac{1}{2}} \tag{19}$$

where Q = 3/2 or 1/2 depending on whether there is or is not an extra upper loop in the group of rays under consideration. Figure 13 is a plot

of Equation 19. As indicated by Brekhovskikh[9] the caustic occurs where $dR/d\theta_A = 0$, on the branch of the curve with the minus sign. Now:

$$dR/d\theta_A = dR/d\varphi = Q \cdot S_U + N \cdot S_f - K^{\frac{1}{2}}/2\varphi^{\frac{1}{2}} \tag{20}$$

Therefore, at the caustic

$$\varphi = K/4(Q \cdot S_U + N \cdot S_f)^2 \tag{21}$$

Differentiating Equation 20, we have

$$d^2R/d\theta_A^2 = K^{\frac{1}{2}}\varphi^{-3/2}/4 \tag{22}$$

Evaluating Equation 22 at the caustic by substitution of Equation 21, we find that

$$(d^2R/d\theta_A^2) = 2(Q \cdot S_U + N \cdot S_f)^3/K \tag{23}$$

IX. Calculation of New Axial Angles of a Ray that Propagates from one Profile to Another

Milder[10] has shown that if the change from one profile to another is sufficiently gradual, there is an invariant called the characteristic time. This invariant can be calculated by the equation:

$$J = (T - X \cdot \cos\theta_A/c_A)/2\pi \tag{24}$$

where J is the characteristic time, T is the full cycle travel time, X is the full cycle range, θ_A is the axial angle of the ray and c_A is the speed of sound at the axis. The conditions on the horizontal gradient for validity are given in detail by Milder for both wave and ray theory. In ocean sound channels a horizontal gradient as small as .03 m/sec/km is safe.

Figure 15 is a plot of characteristic time vs axial angle for the

profiles that we have been considering. To find the angle in one profile that is equivalent to an angle in another one finds the value J corresponding to the angle, θ_{A_1} , in the first profile moves horizontally to the curve for the second profile and under the same J one finds the value, θ_{A_2} , in the second profile.

X. Calculation of Range Annotated Ray Angle Diagrams

Flatte[11] and Cox[12] have discussed the range annotated ray angle diagram and its applications. A program (for the pocket programmable calculator) adapted to generating data for such a diagram is included in the supplement. The depth difference Z from the reference level and the angle θ are computed from the range on a segment shorter than that from reference level to vertex. Longer paths are plotted by symmetry and addition. There are two cases: One where the given range is R_{oz} in Equation 8, the other where the given range is R_{zz_v} in Equation 7. In the first case:

$$I_1 = R_{oz}\ \beta(\tan\theta_o)/B_1 Z_v \tag{25}$$

In the second case:

$$I_1 = 1 - R_{zz_v}\ \beta(\tan\theta_o)/B_1 Z_v \tag{26}$$

The value of I_1 is used to obtain X from the I_x tables of the incomplete beta-function. Then

$$Z = Z_v X^{1/\beta} \tag{27}$$

and from Equation 1

$$c/c_o = (1 - |\alpha Z|^{\beta})^{-1/2} \tag{28}$$

but

$$\theta = \arccos\left(c \cos\theta_0 / c_0\right)$$

or

$$\theta = \arccos\left(\cos\theta_0 \Big/ \sqrt{1 - |\alpha z|^{\beta}}\right) \tag{29}$$

XI. Notes on the Values of β in Asymmetric Profiles Based on the Hirsch-Carter Model

Hirsch and Carter have pointed out that, in symmetric models of the near axis sound transmission, the observed time dispersion of arrivals occurs only in that subset of the β family for which $1 < \beta < 2$. The actual sound channel, however, is grossly asymmetrical. Because the refraction below the axis is so much weaker than that above, rays at more than a very small axial angle will spend much more time below the axis than above it so that the overall dispersion pattern is like that of a symmetric channel with β near the below axis value of 1.25 or 1.26, the profiles above the axis are fitted, however, with values of β between 2 and 3. This would tend to reduce the dispersion below what one would get by reflection of the lower half of the channel.

XII. Acknowledgements

This work was supported by ONR Contract N00014-74-C-0262; NR 083-004. John C. Beckerle suggested the use of the Hirsch-Carter profile and did some preliminary calculations with it. P. Hirsch suggested use of the incomplete beta-function for computing ray segments. Additional helpful discussions with J. C. Beckerle, Earl E. Hays and George V. Frisk, are gratefully acknowledged.

*Supplement appended to this report

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CAPTIONS TO FIGURES

Figure 1 Path of a ray from reference level to vertex within a layer. Calculation of range and travel time of the segment from O to Z or that from Z to Z_V is discussed in the text. The general segment Q to S can be expressed as a difference of segments of either of the above types.

Figure 2 Geometry of sound speed profiles described by Equation 1A. C_0=1.49275 in all the curves. The other parameters are listed after the indicated number of each curve as follows:

1. $\alpha = 10^{-28}$, $\beta = .05$;
2. $\alpha = 10^{-14}$, $\beta = .10$;
3. $\alpha = 3\times10^{-6}$, $\beta = .25$;
4. $\alpha = 5.6\times10^{-5}$, $\beta = .33$;
5. $\alpha = 1.6\times10^{-3}$, $\beta = .50$;
6. $\alpha = 7.0\times10^{-3}$, $\beta = .65$;
7. $\alpha = .04$, $\beta = 1.0$;
8. $\alpha = .117$, $\beta = 1.5$;
9. $\alpha = .20$, $\beta = 2.0$;
10. $\alpha = .275$, $\beta = 2.5$;
11. $\alpha = .343$, $\beta = 3.0$.

Figure 3 Slopes of the sound speed profiles of Figure 2 on a logarithmic plot.

Figure 4 A sound speed profile from the center of a cold ring eddy is indicated by the continuous line. Two layers according to Equation 1 have been fitted at the axis and the points marked by X. The calculated sound speeds at other points are indicated by dots. The parameters are: $\alpha_U = 0.428668$, $\beta_U = 2.87968$, $c_0 = 1.48867$, $\alpha_L = .0325903$, and $\beta_L = 1.25083$

Figure 5 A sound speed profile in the Sargasso Sea outside of the eddy is indicated by the continuous line. Two layers according to Equation 1 have been fitted at the axis and at the points marked by X. The calculated sound speeds at other points are indicated by dots. The parameters are: $\alpha_U = .335904$ $\beta_U = 2.05506$, $C_0 = 1.49275$, $\alpha_L = .0321452$, and $\beta_L = 1.25931$

Figure 6 The upper portion of the eddy profile, Figure 4, is fitted by two layers. The previous fit is retained below the sound channel axis. The new parameters are as follows:
$\alpha_U = 0.44$, $\beta_U = 2.857$, $(c_0)_U = 1.48867$, $\alpha_M = 0.061$
$\beta_M = 1.0000$, $(c_0)_M = 1.49600$

Figure 7 The upper portion of the Sargasso Sea profile, Figure 5, is fitted by two layers. The previous fit is retained below the sound channel axis. The new parameters are as follows:
$\alpha_U = 0.315$, $\beta_U = 2.000$, $(c_0)_U = 1.49275$, $\alpha_M = 4.01 \times 10^{-15}$,
$\beta_M = 0.100$, $(c_0)_M = 1.4989$; $(z_i)_M$ at the interface $= 0.0216$,
$(z_i)_U = 0.575$

Figure 8 Range and vertex depth vs axial angle for the three layer fit of the eddy profile (Figures 4, 6). Range of a loop above the axis, one below the axis, and a full ray cycle are shown. Vertex depth is for the upper loop and is therefore the shallowest point reached by the ray.

Figure 9 Travel time vs axial angle for the same profile, fit, and paths as Figure 8.

Figure 10 Range and vertex depth vs axial angle for the three layer fit of the Sargasso Sea profile (Figures 5 and 7). The data presented corresponds to that presented in Figure 8. The constancy of range of the upper loop over the axial angles 0-10.4 is a property of the fit with $\beta = 2.0$ as noted in the Hirsch-Carter[1] paper.

Figure 11 Travel time vs axial angle for the same profile, fit, and paths as Figure 10.

Figure 12 Range vs axial angle of high order rays in the Sargasso Sea profile (Figures 5 and 7). The order is, of course, the number of loops below the axis. The receiver depth is .85 km. The source is on the axis. There are four rays belonging to each order. This figure may be used as described in the text to find axial angles of eigen rays at a given range.

Figure 13 Range vs axial angle for two rays of the 17th order. This figure demonstrates the formation of a caustic as discussed in the text.

Figure 14 Range vs axial angle of a ray segment from vertex to receiver depth (.85 km, i.e. .4 km above the axis) in the Sargasso Sea profile (Figures 5 and 7). The solid line shows values calculated using the Hirsch-Carter model with tables of the incomplete beta-function. The circles show values from the parabolic fit, $r_x^2 = K(\theta_A - (\theta_A)_p)$, with $K = 2.9781$ and $(\theta_A)_p = 6.892°$

Figure 15 Characteristic time vs axial angle. This figure can be used for estimating the change in axial angle when a ray propagates through a transition region from one sound velocity profile to another.

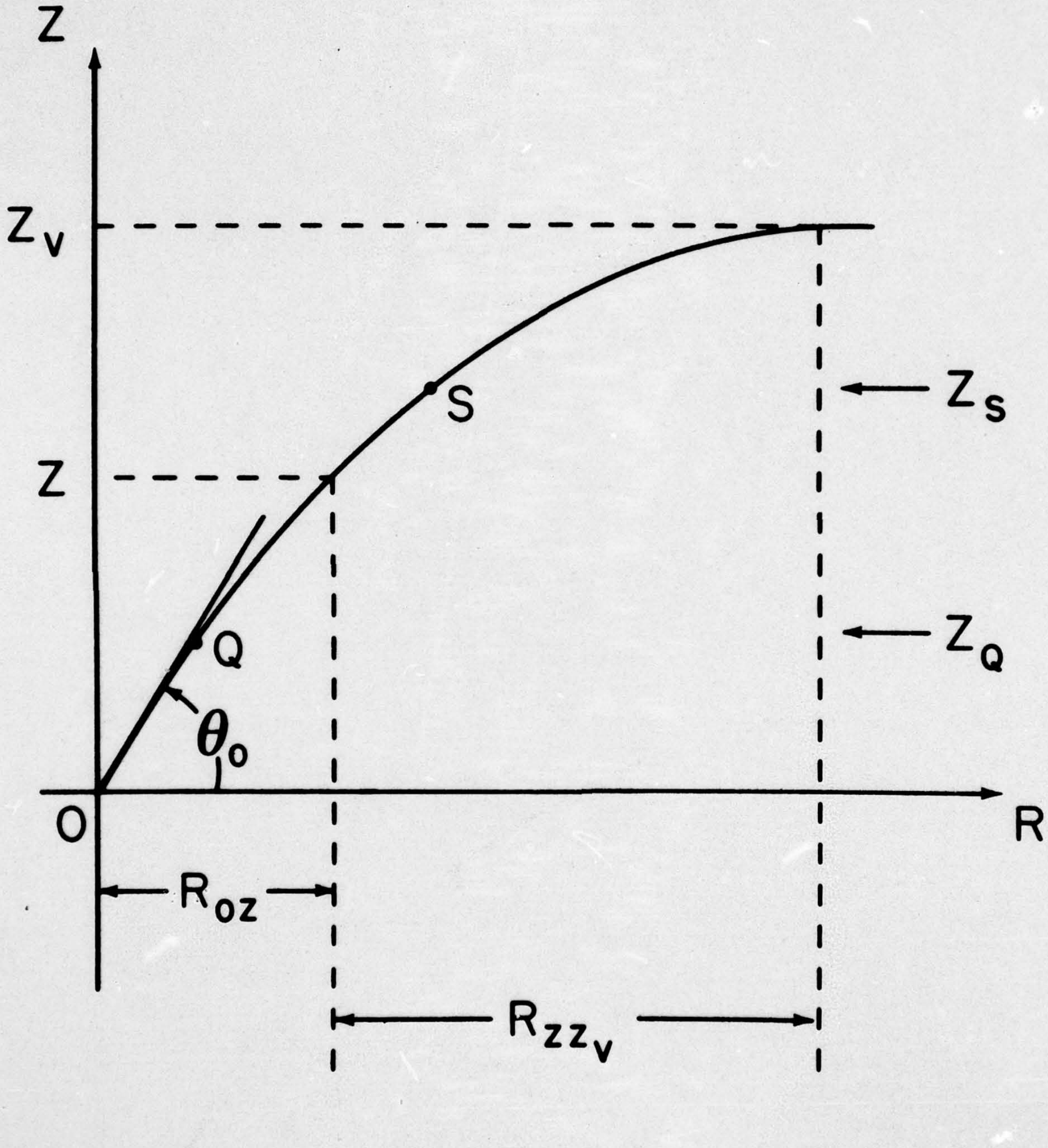


Fig. 1

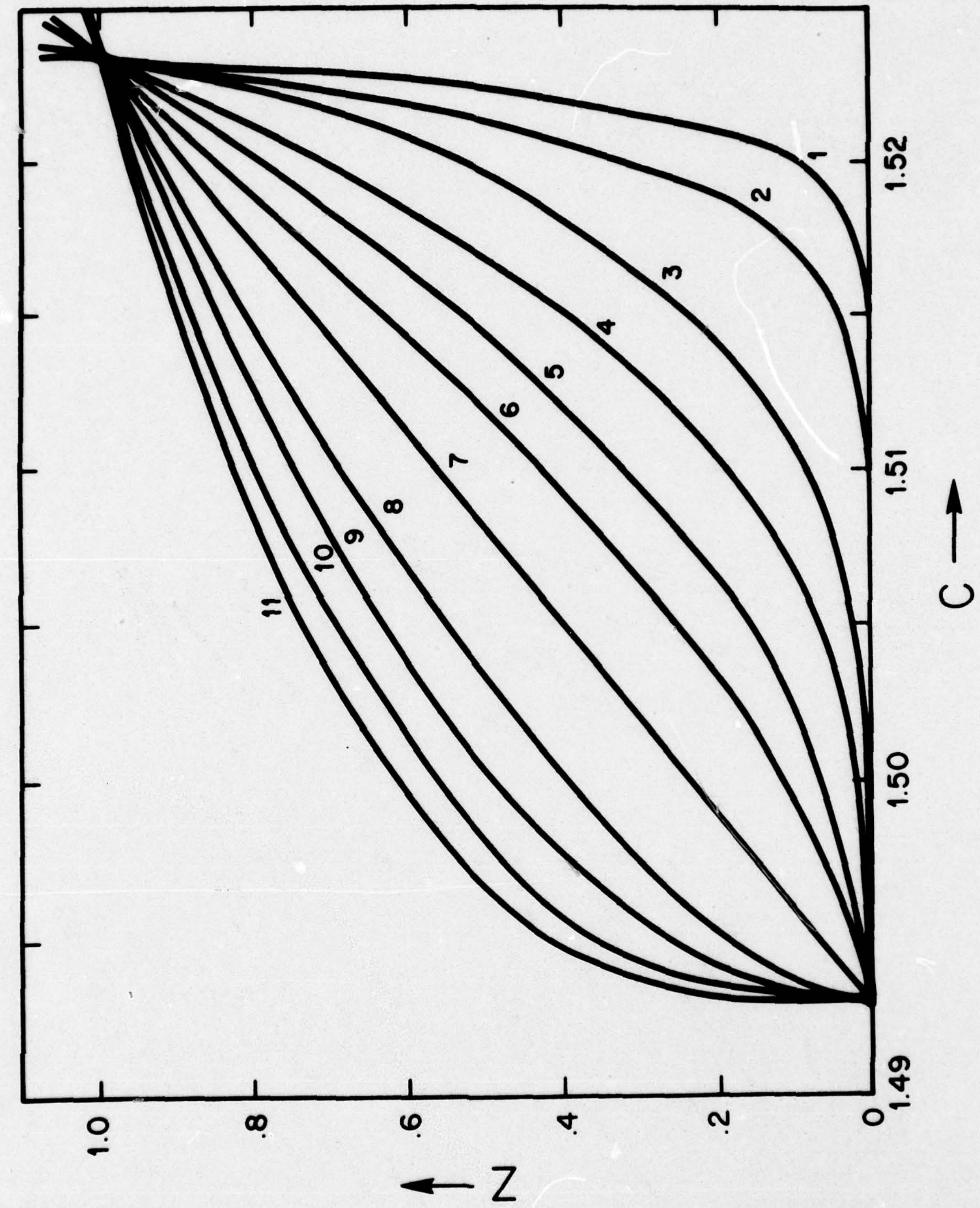


Fig. 2

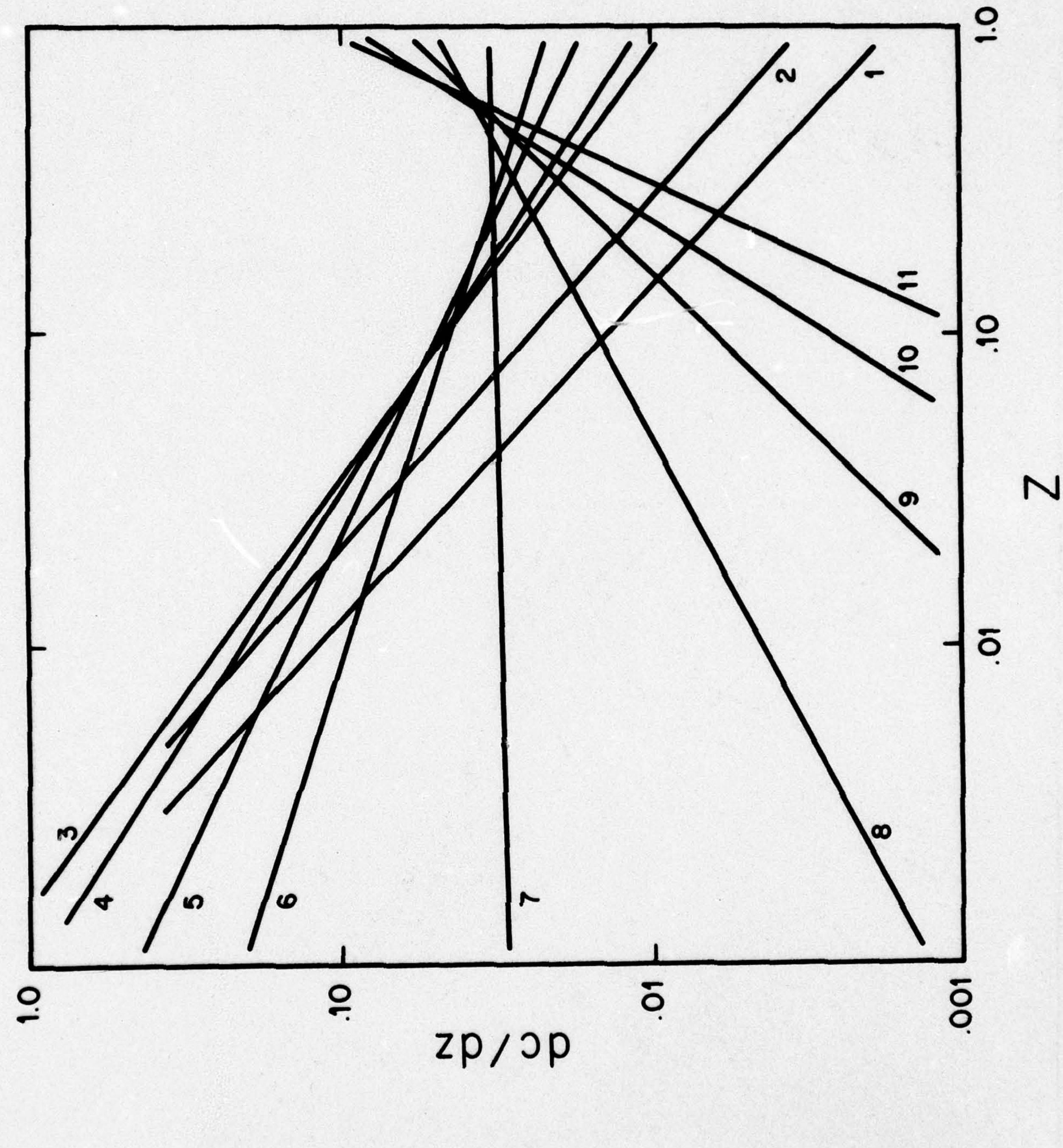


Fig. 3

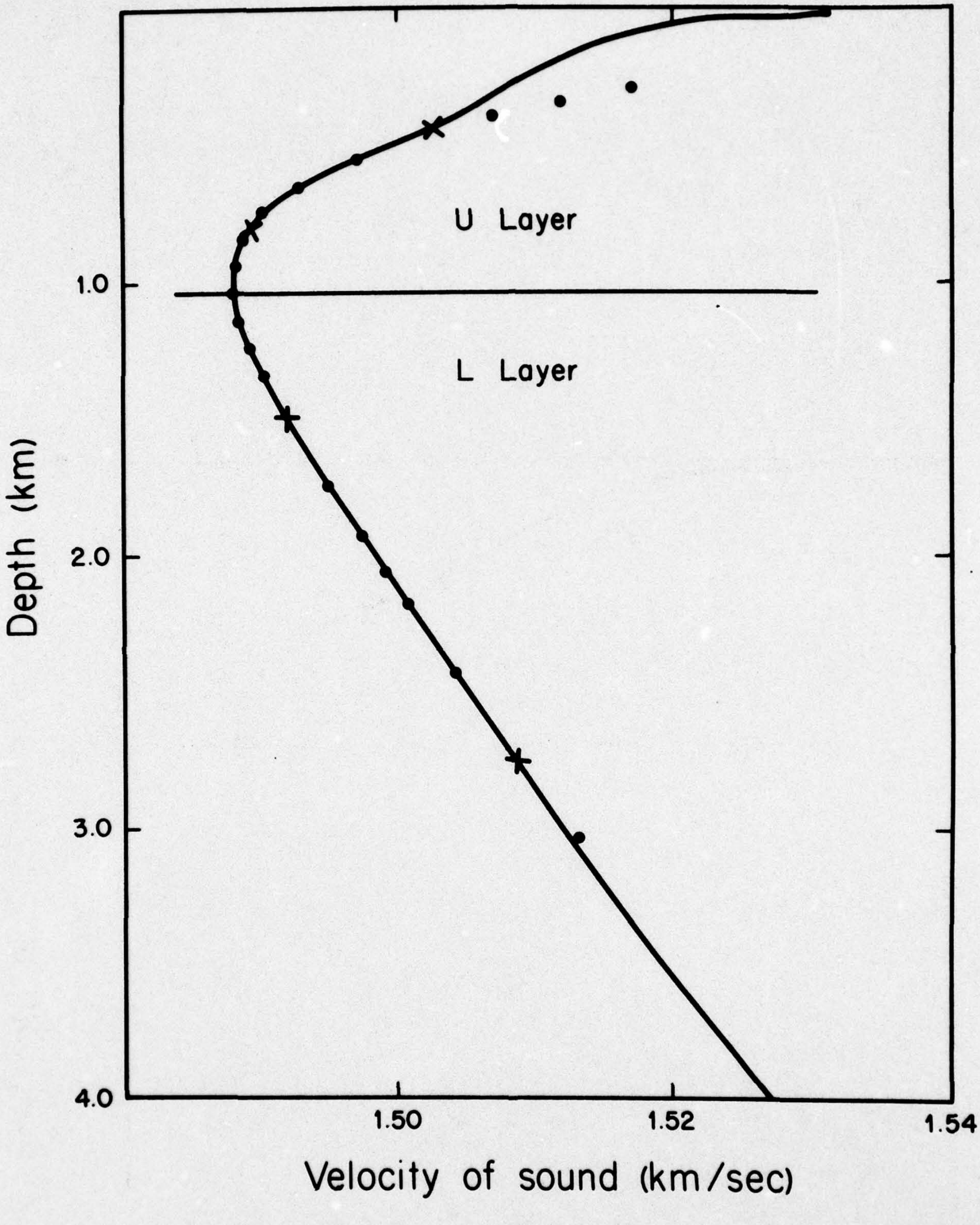

U Layer
L Layer
Depth (km)
1.0
2.0
3.0
4.0
1.50
1.52
1.54
Velocity of sound (km/sec)

Fig. 4

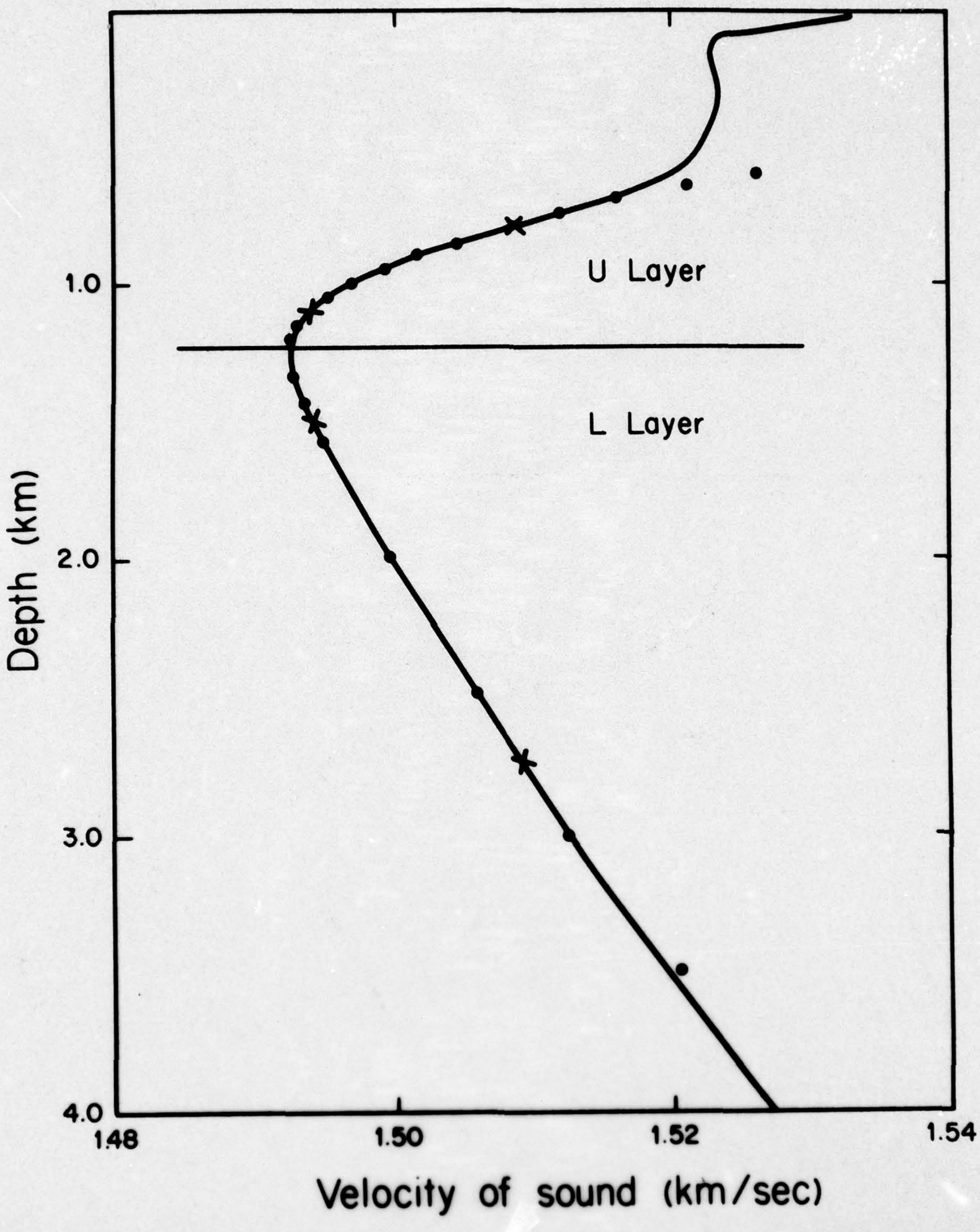
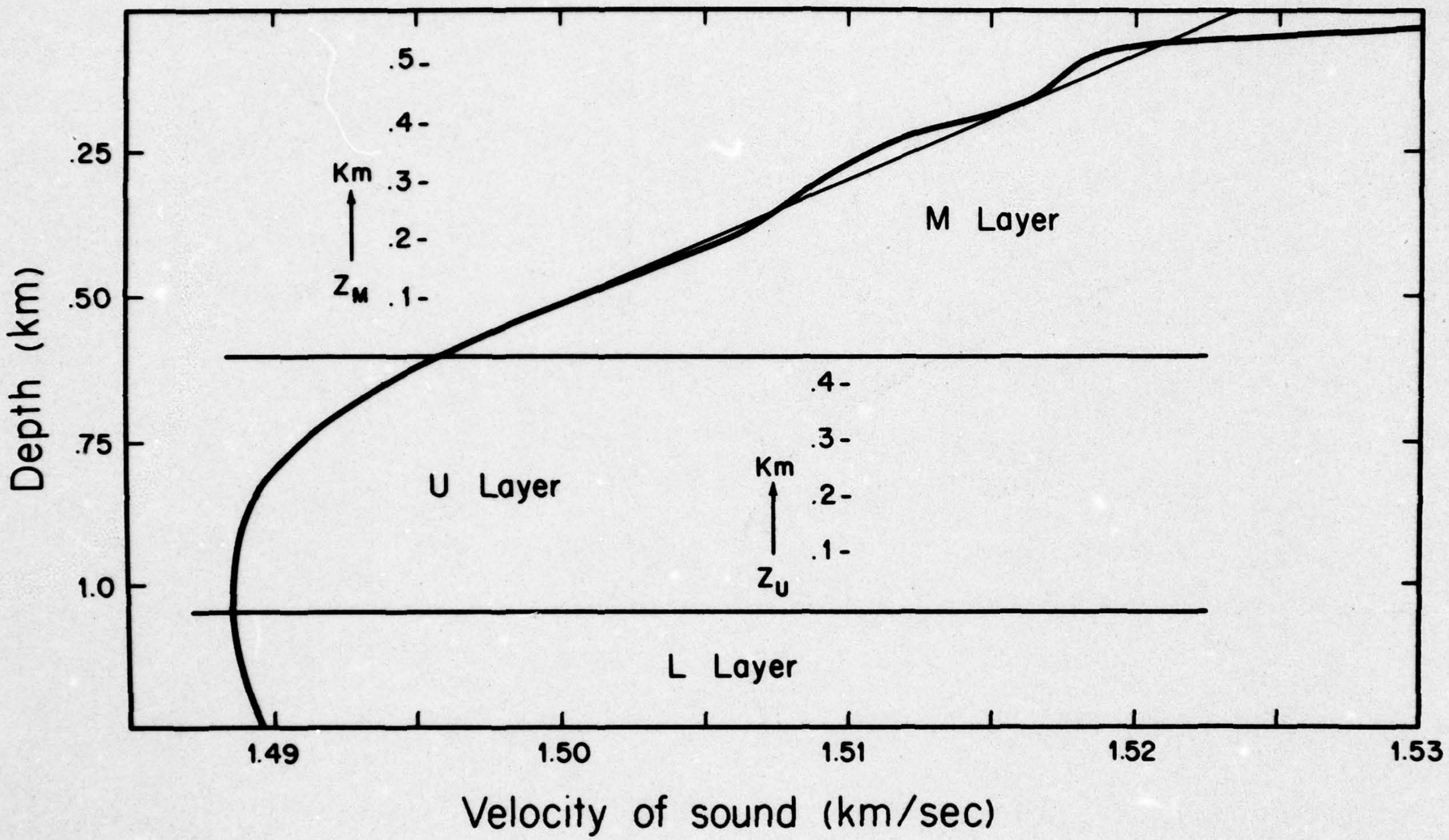

U Layer
L Layer
Depth (km)
1.0
2.0
3.0
4.0
1.48
1.50
1.52
1.54
Velocity of sound (km/sec)

Fig. 5



Depth (km)
.25
.50
.75
1.0
Velocity of sound (km/sec)
1.49
1.50
1.51
1.52
1.53
M Layer
U Layer
L Layer
Km
Z_M
.1-
.2-
.3-
.4-
.5-
Km
Z_U
.1-
.2-
.3-
.4-

Fig. 6

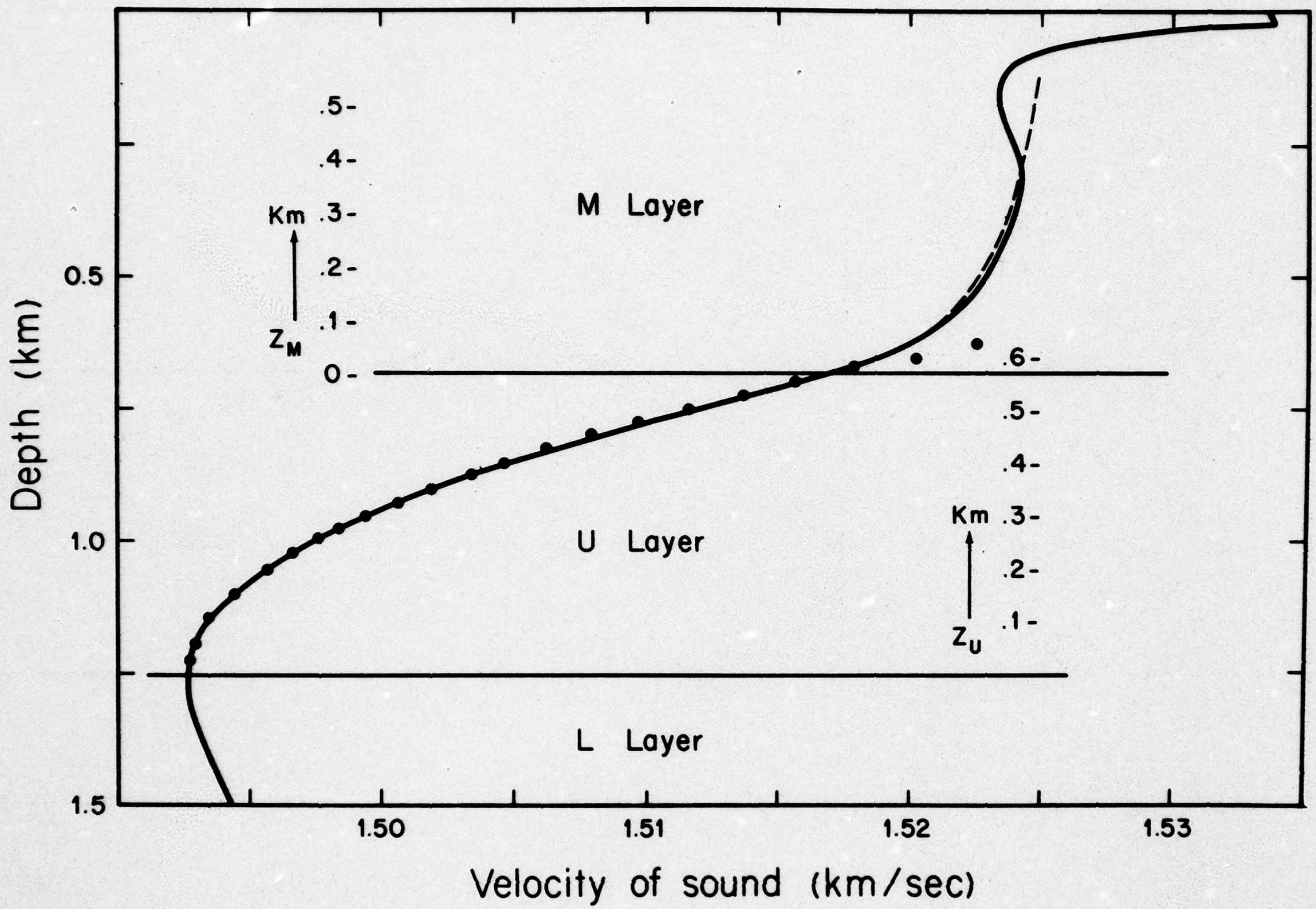

Depth (km)
0.5
1.0
1.5
Velocity of sound (km/sec)
1.50
1.51
1.52
1.53
M Layer
U Layer
L Layer
Km
Z_M
.5-
.4-
.3-
.2-
.1-
0-
Km
Z_U
.6-
.5-
.4-
.3-
.2-
.1-

Fig. 7

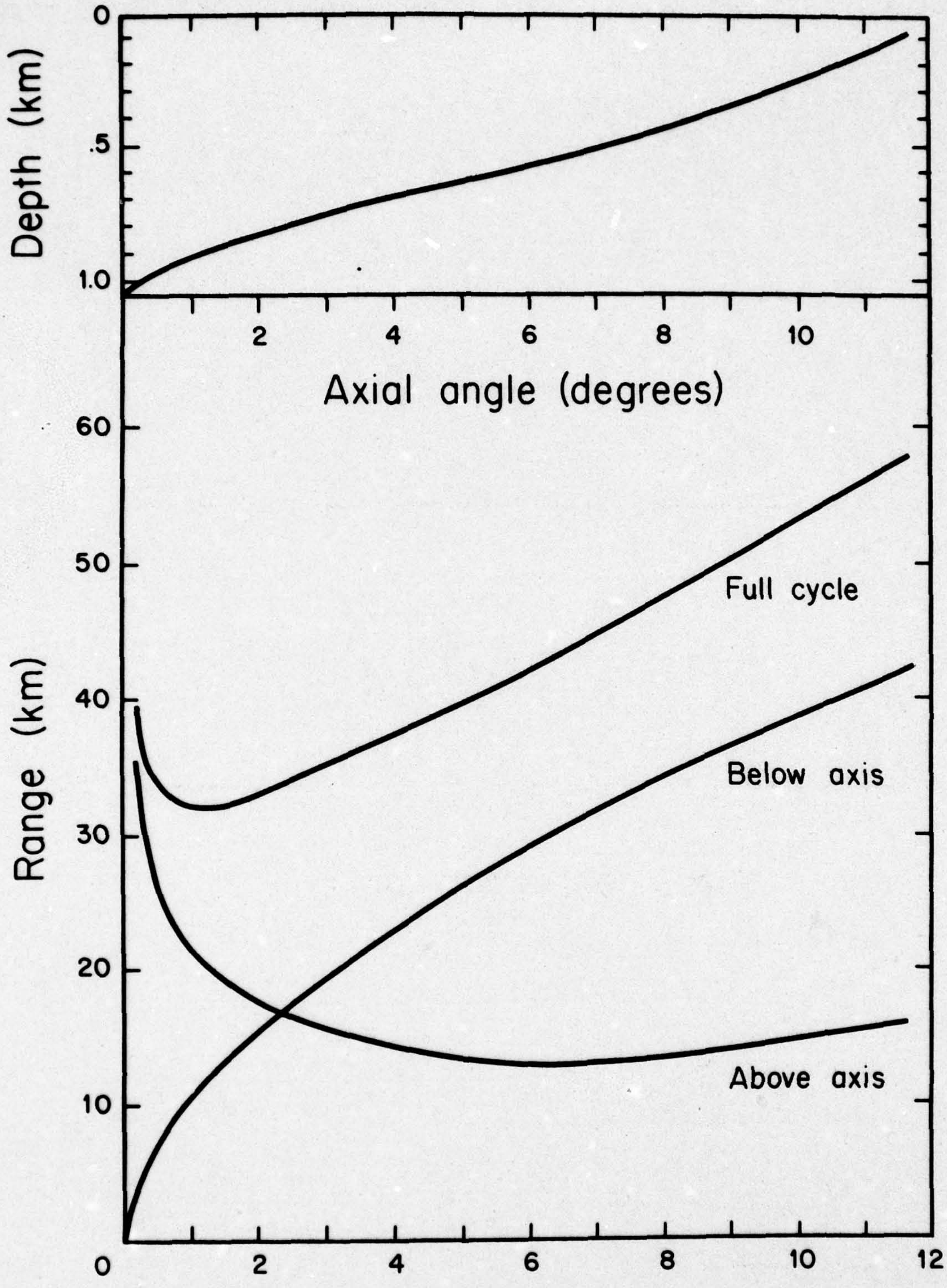
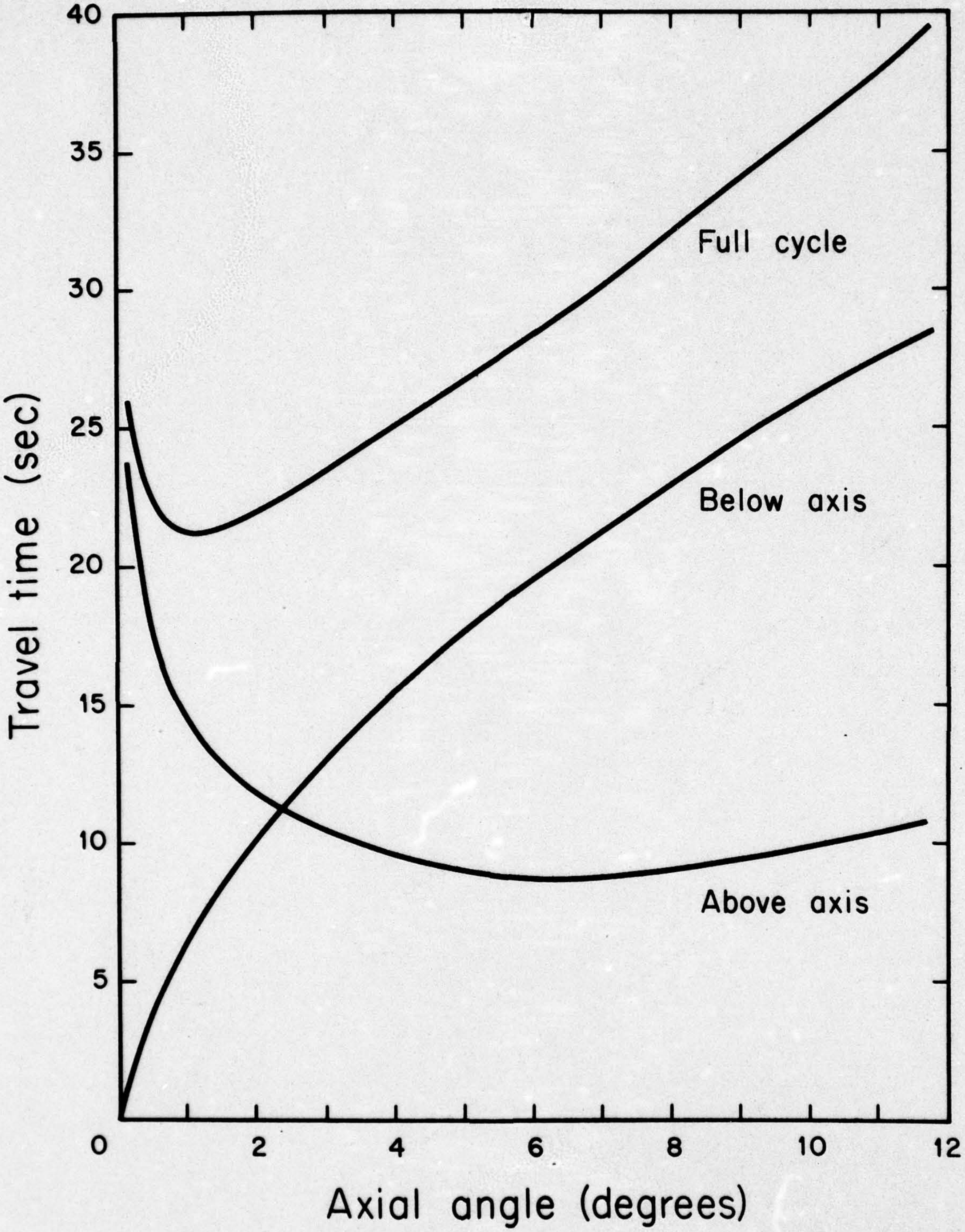
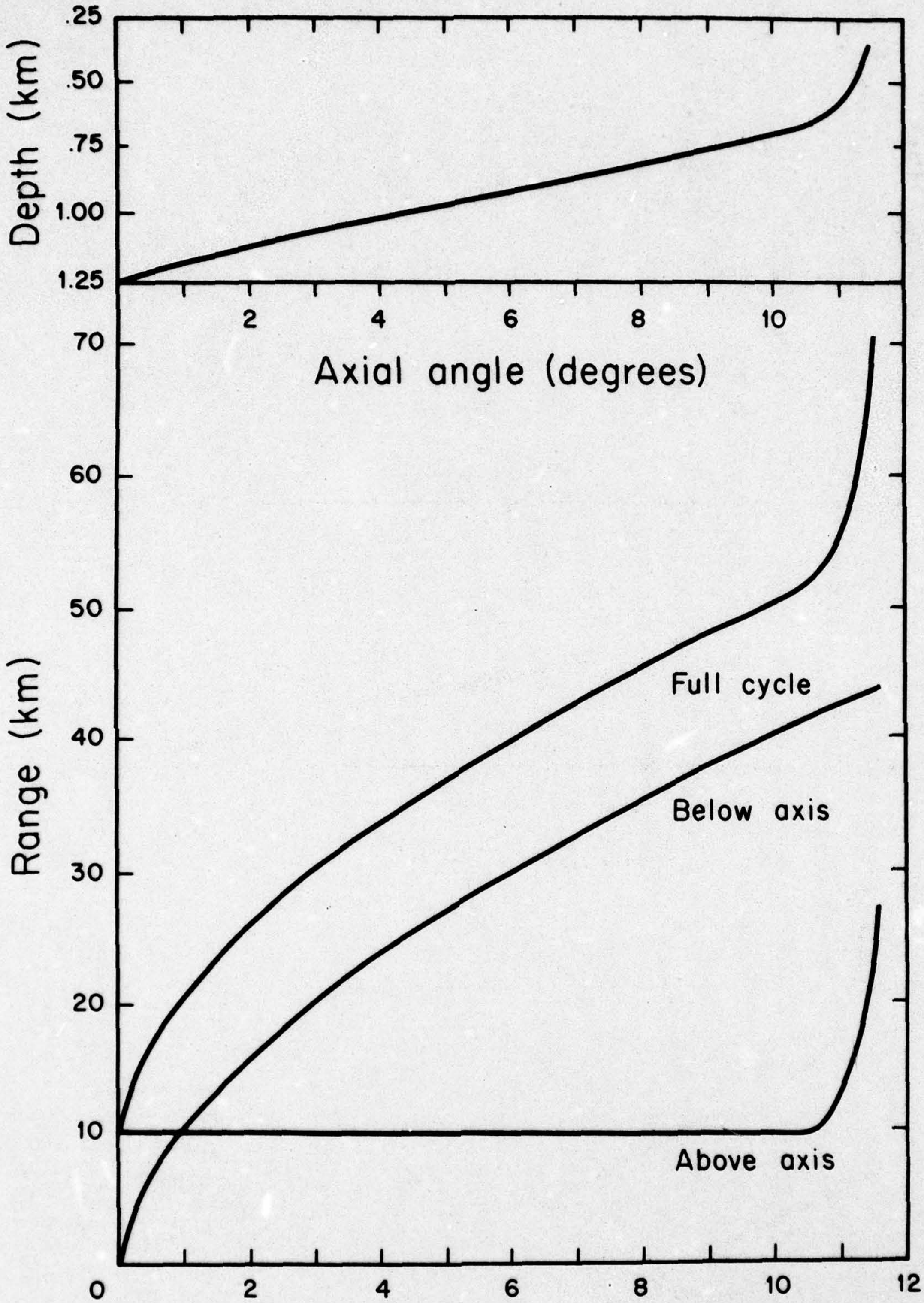

0
.5
1.0
Depth (km)
2
4
6
8
10
Axial angle (degrees)
60
50
40
30
20
10
0
Range (km)
Full cycle
Below axis
Above axis
2
4
6
8
10
12

Fig. 8



40
35
30
25
20
15
10
5
0
2
4
6
8
10
12
Travel time (sec)
Axial angle (degrees)
Full cycle
Below axis
Above axis

Fig. 9



Depth (km)
.25
.50
.75
1.00
1.25
2
4
6
8
10
Axial angle (degrees)
Range (km)
70
60
50
40
30
20
10
0
Full cycle
Below axis
Above axis
12

Fig. 10

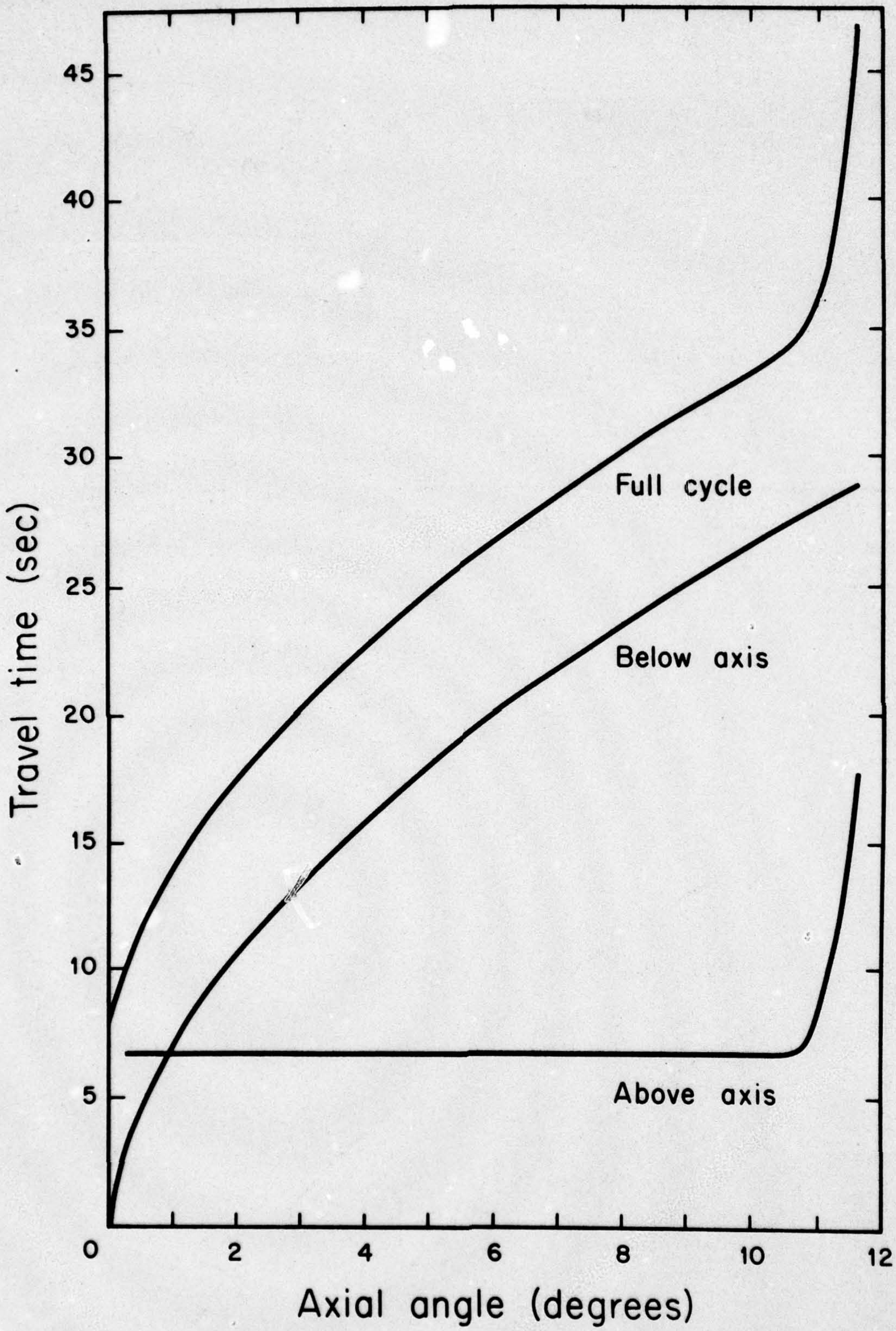

45
40
35
30
25
20
15
10
5
0
2
4
6
8
10
12
Travel time (sec)
Axial angle (degrees)
Full cycle
Below axis
Above axis

Fig. 11

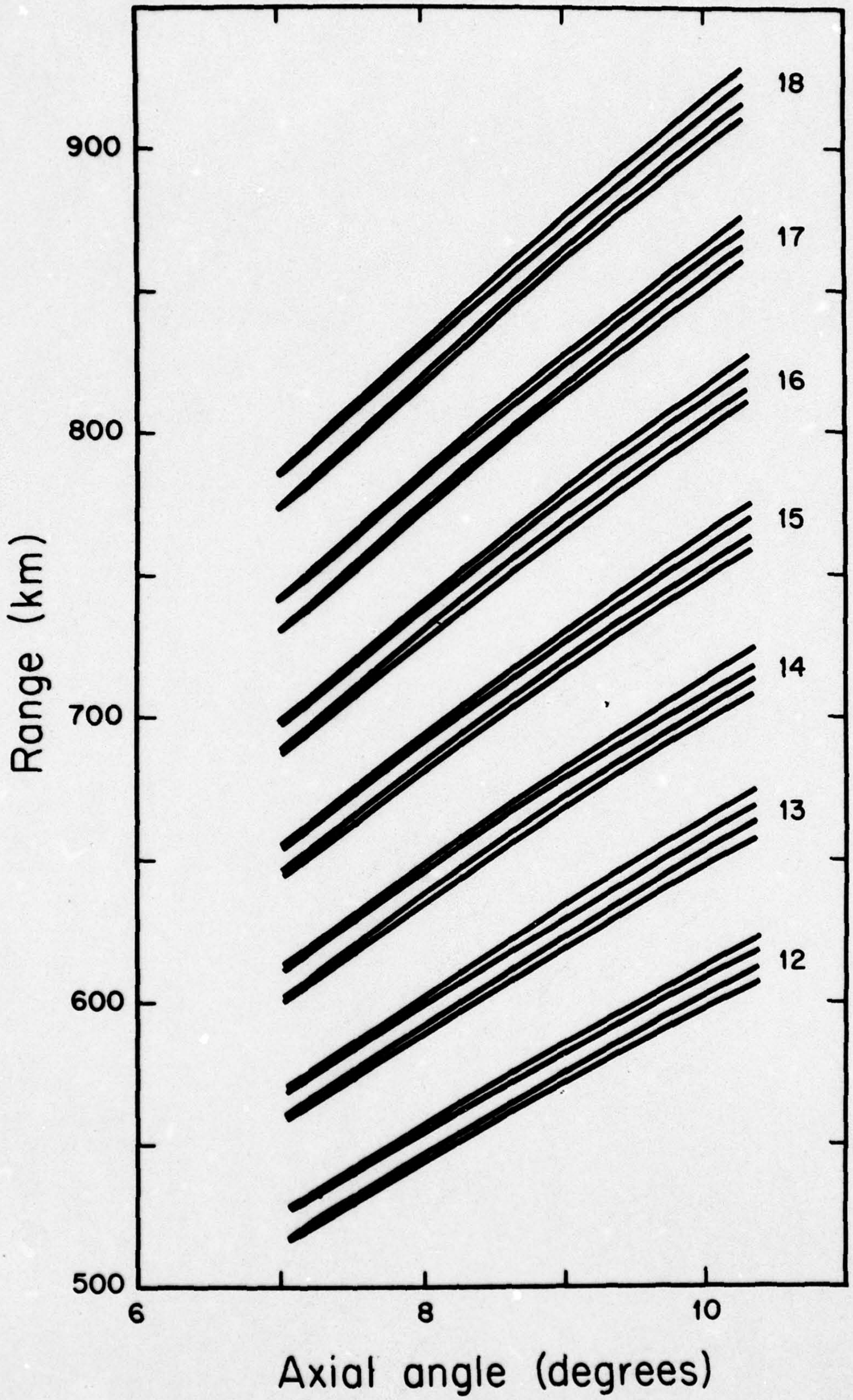

18
17
16
15
14
13
12
900
800
700
600
500
Range (km)
6
8
10
Axial angle (degrees)

Fig. 12

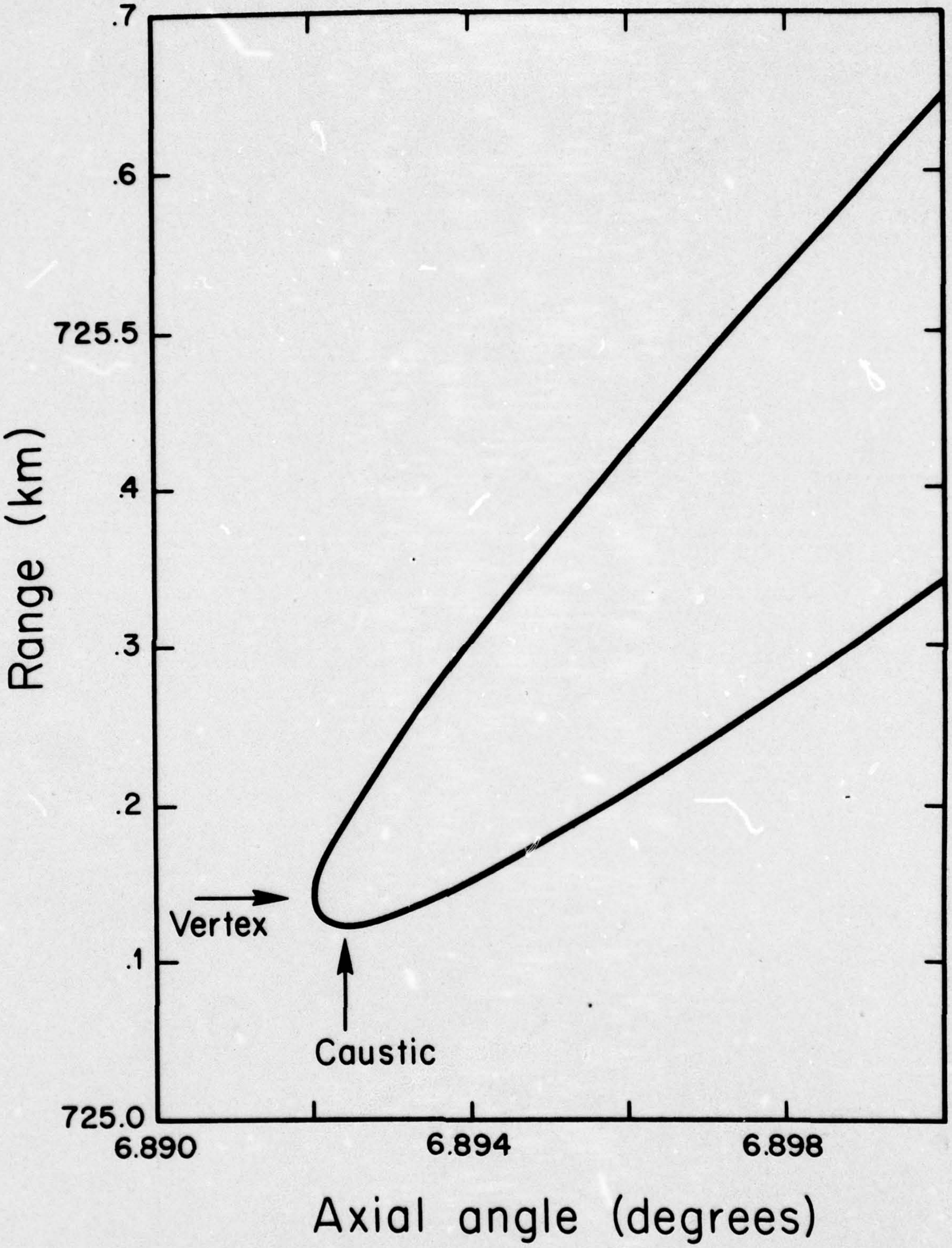

.7
.6
725.5
.4
.3
.2
.1
725.0
Range (km)
Vertex
Caustic
6.890
6.894
6.898
Axial angle (degrees)

Fig. 13

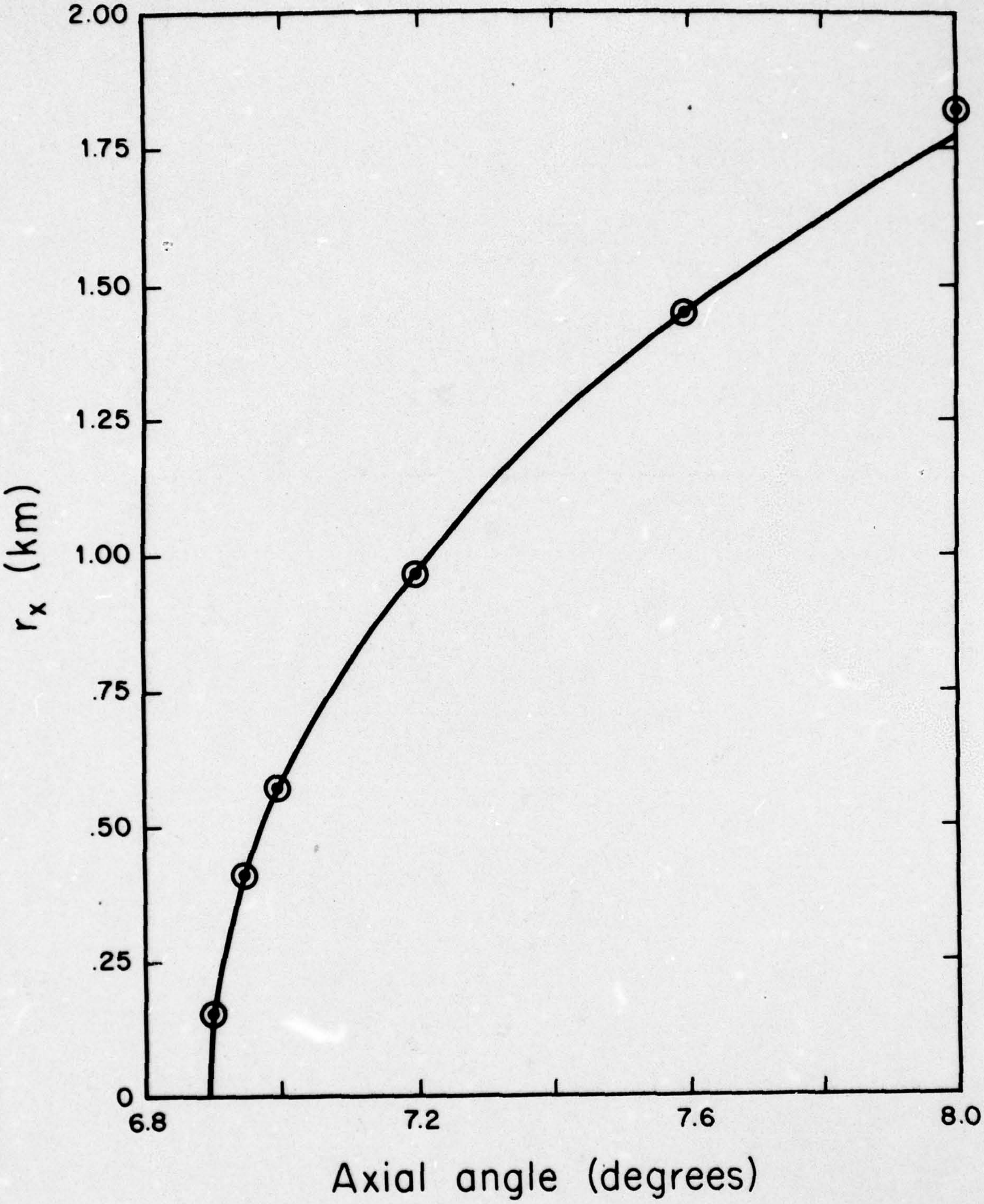

2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0
6.8
7.2
7.6
8.0
r_x (km)
Axial angle (degrees)

Fig. 14

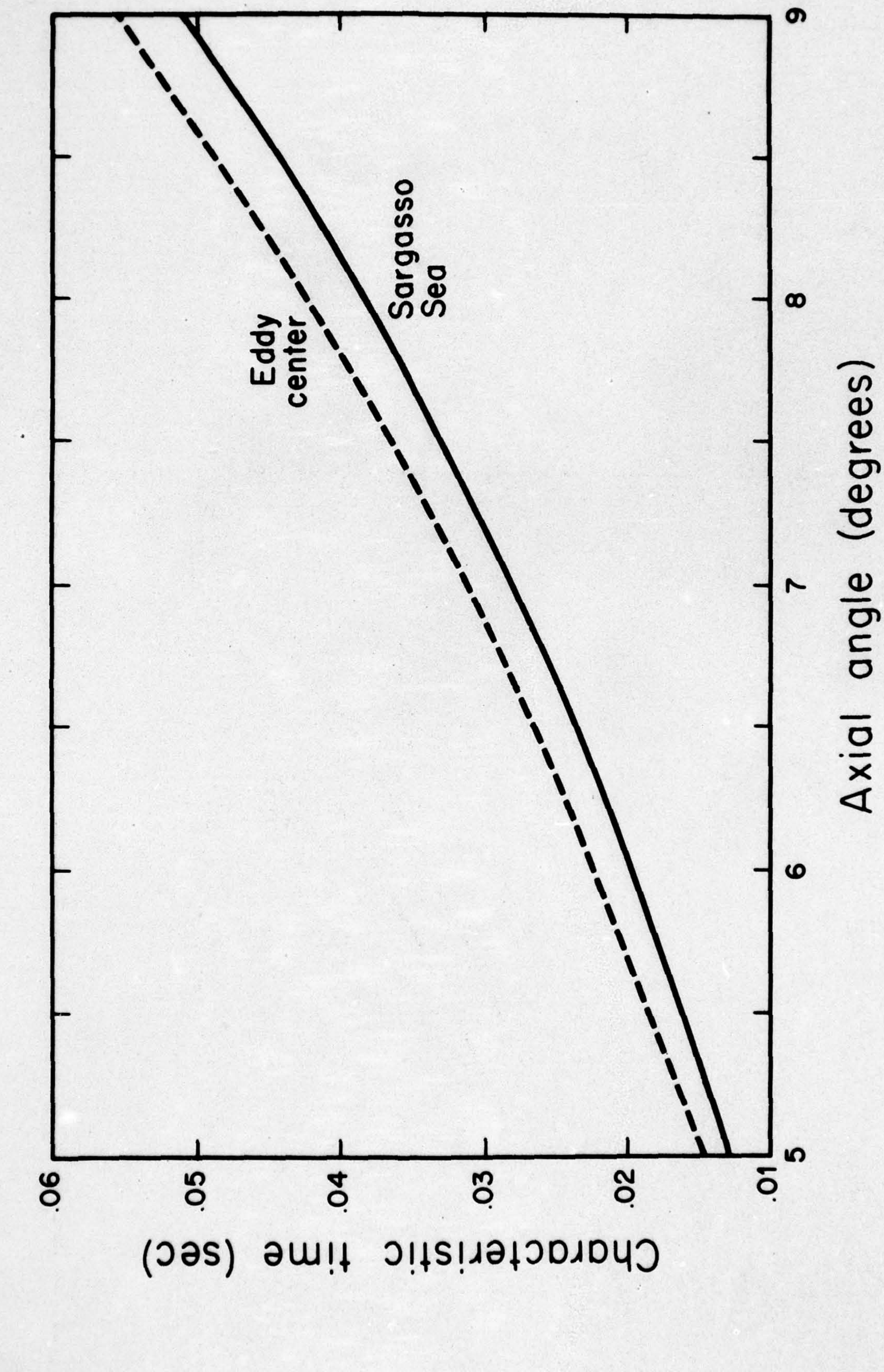

Characteristic time (sec)
.06
.05
.04
.03
.02
.01
5
6
7
8
9
Axial angle (degrees)
Eddy center
Sargasso Sea

Fig. 15

Supplement to

RAY CALCULATIONS OF OCEAN SOUND CHANNELS USING A POCKET PROGRAMMABLE CALCULATOR AND EXTENDED FORMS OF THE HIRSCH-CARTER MATHEMATICAL MODEL WITH TABLES OF THE INCOMPLETE BETA-FUNCTION

L. Baxter, II

A. Tables of the Incomplete Beta-function

The complete beta-function, $B(p,q)$ of the variables p and q is given by

$$B(p,q) = \Gamma(p)\cdot\Gamma(q) / \Gamma(p+q) \tag{1}$$

or alternatively by:

$$B(p,q) = \int_0^1 x^{(p-1)}(1-x)^{(q-1)}\,dx \tag{2}$$

The incomplete beta-function, $B_x(p,q)$ is given by

$$B_x(p,q) = \int_0^x y^{(p-1)}(1-y)^{(q-1)}\,dy \quad , \quad 0<x<1 \tag{3}$$

The function $I_x(p,q)$ sometimes called the relative incomplete beta-function is given by

$$I_x(p,q) = B_x(p,q) / B(p,q) \tag{4}$$

In these tables and in those of Pearson[1] I_x is given as a function of p, q and x, while $B(p,q)$ is tabulated at the top of each column of $I_x(p,q)$. In the present tables q is taken equal to .5, the only value it assumes in the Hirsch-Carter model equations.

In the paper to which this supplement is appended the Hirsch-Carter model equation is written

$$c^2 = c_0^2\left(1 - |\alpha z|^{\beta}\right)^{-1} \tag{5}$$

and it is shown that range and travel time can be written for sound ray segments in terms of the following quantities

$$B_1 = B(1/\beta, .5) \tag{6}$$

$$I_1 = I_x(1/\beta, .5) \tag{7}$$

$$B_2 = B(1+1/\beta, .5) \tag{8}$$

$$I_2 = I_x(1+1/\beta, .5) \tag{9}$$

where $$x = (Z/Z_v)^\beta \tag{10}$$

and Z_v is the value of Z for which a sound ray vertexes (i.e. becomes horizontal at its maximum or minimum depth of excursion). To use these tables for sound ray calculations, B_1 is taken from the top of the column with $p = 1/\beta$; I_1 is taken opposite x from the same column; B_2 is taken from the top of the column with $p = 1 + 1/\beta$; and I_2 is taken opposite x from that column.

For some computations it may be necessary to enter the tables with a value of I_1 and interpolate to find a value of x .

The $I_x(p, q)$ Function

$q = 0.5$ $p = .20$ to $.30$

x	p = .2	.25	.30
B(p,q)	6.268655		4.554444
.01	.3178035		.184054
.02	.3653681		.2268620
.03	.3965678		.2565072
.04	.4204138		.2799599
.05	.4399803		.2997001
.06	.4567164		.3169302
.07	.4714288		.3323333
.08	.4846165		.3463399
.09	.4966091		.3592383
.10	.5076395		.3712348
.11	.5178767		.3824811
.12	.5274482		.3930920
.13	.5364528		.4031581
.14	.5449675		.4127504
.15	.5530558		.4219277
.16	.5607692		.4307382
.17	.5681497		.4392212
.18	.5752329		.4474108
.19	.5820503		.4553367
.20	.5886254		.4630221
.21	.5949836		.4704901
.22	.6011419		.4777586
.23	.6071190		.4848451
.24	.6129291		.4917640
.25	.6185856		.4985279
.26	.6241008		.5051492
.27	.6294845		.5116379
.28	.6347481		.5180048
.29	.6398979		.5242568
.30	.6449436		.5304035
.31	.6498911		.5364509
.32	.6547480		.5424062
.33	.6595198		.5482757
.34	.6642126		.5540657
.35	.6688304		.5597800
.36	.6733793		.5654249
.37	.6778630		.5710042
.38	.6822851		.5765225
.39	.6866506		.5819842
.40	.6909627		.5873926
.41	.6952244		.5927516
.42	.6994404		.5980653
.43	.7036116		.6033359
.44	.7077423		.6085674
.45	.7118349		.6137625
.46	.7158926		.6189235
.47	.7199167		.6240546
.48	.7239106		.6291571
.49	.7278763		.6342343
.50	.7318162		.6392893

x	p = .2	.25	.30
	6.268655		4.554444
.51	.7357336		.6443241
.52	.7396286		.6493406
.53	.7435044		.6543426
.54	.7473620		.6593315
.55	.7512063		.6643100
.56	.7550366		.6692804
.57	.7588556		.6742451
.58	.7626655		.6792067
.59	.7664682		.6841673
.60	.7702656		.6891297
.61	.7740599		.6940961
.62	.7778531		.6990691
.63	.7816472		.7040516
.64	.7854444		.7090461
.65	.7892470		.7140552
.66	.7930571		.7190821
.67	.7968770		.7241296
.68	.8007092		.7292010
.69	.8045565		.7342995
.70	.8084209		.7394283
.71	.8123058		.7445919
.72	.8162140		.7497936
.73	.8201486		.7550378
.74	.8241130		.7603290
.75	.8281112		.7656721
.76	.8321468		.7710727
.77	.8362240		.7765365
.78	.8403481		.7820698
.79	.8445238		.7876795
.80	.8487573		.7933743
.81	.8530544		.7991621
.82	.8574232		.8050534
.83	.8618715		.8110591
.84	.8664085		.8171923
.85	.8710454		.8234678
.86	.8757945		.8299029
.87	.8806710		.8365184
.88	.8856927		.8433383
.89	.8908808		.8503926
.90	.8962620		.8577176
.91	.9018698		.8653592
.92	.9077468		.8733766
.93	.9139497		.8818481
.94	.9205569		.8908814
.95	.9276810		.9006315
.96	.9354948		.9113368
.97	.9442899		.9233995
.98	.9546366		.9376046
.99	.9620097		.9559839
1.00	1.000000		1.0000000

$B = 4.0566228$

A	B
.350000	.5000000
N	K
25	9
X	I
.000100	.0280396
.000200	.0357387
.000300	.0411886
.000400	.0455524
.000500	.0492533
.000600	.0524994
.000700	.0554103
.000800	.0580623
.000900	.0605066
A	B
.350000	.5000000
N	K
25	9
X	I
.001000	.0627804
.002000	.0800277
.003000	.0922420
.004000	.1020267
.005000	.1103288
.006000	.1176139
.007000	.1241499
.008000	.1301070
.009000	.1356003

The $I_x(p,q)$ Function

$q=0.5$ $p=.35$ to $.45$

x	p= .35	.40	.45	x	p= .35	.40	.45
B(p,q)	4.0566228	3.6790923	3.3820539		4.0566228	3.6790923	3.3820539
.01	.1407124	.1078504	.0828482	.51	.6050637	.5692794	.5365068
.02	.1795816	.1425152	.1133517	.52	.6105126	.5750914	.5426226
.03	.2072370	.1678532	.1362559	.53	.6159500	.5808963	.5487374
.04	.2294953	.1886002	.1553352	.54	.6213784	.5866976	.5548538
.05	.2484703	.2065132	.1720190	.55	.6268009	.5924978	.5609750
.06	.2652006	.2224676	.1870295	.56	.6322194	.5982989	.5671022
.07	.2802834	.2369720	.2007909	.57	.6376363	.6041037	.5732393
.08	.2940971	.2503520	.2135768	.58	.6430547	.6099150	.5793985
.09	.3068985	.2628293	.2255753	.59	.6484768	.6157355	.5855528
.10	.3188713	.2745643	.2369230	.60	.6539054	.6215677	.5917344
.11	.3301514	.2856761	.2477220	.61	.6593431	.6274146	.5979367
.12	.3408430	.2962561	.2580513	.62	.6647923	.6332787	.6041629
.13	.3510282	.3063771	.2679737	.63	.6702564	.6391632	.6104158
.14	.3607711	.3160962	.2775390	.64	.6757377	.6450714	.6166986
.15	.3701264	.3254619	.2867891	.65	.6812396	.6510065	.6230150
.16	.3791373	.3345131	.2957588	.66	.6867654	.6569715	.6293678
.17	.3878407	.3432824	.3044766	.67	.6923178	.6629704	.6357617
.18	.3962678	.3517990	.3129677	.68	.6979008	.6690063	.6422001
.19	.4044461	.3600866	.3212536	.69	.7035174	.6750832	.6486871
.20	.4123976	.3681658	.3293527	.70	.7091722	.6812059	.6552272
.21	.4201434	.3760556	.3372816	.71	.7148689	.6873782	.6618254
.22	.4277001	.3837714	.3450539	.72	.7206117	.6936052	.6684865
.23	.4350841	.3913282	.3526835	.73	.7264055	.6998915	.6752161
.24	.4423097	.3987384	.3601815	.74	.7322553	.7062432	.6820199
.25	.4493878	.4060128	.3675573	.75	.7381666	.7126657	.6889045
.26	.4563307	.4131626	.3748212	.76	.7441452	.7191660	.6958767
.27	.4631477	.4201962	.3819810	.77	.7501978	.7257510	.7029445
.28	.4698485	.4271229	.3890450	.78	.7563317	.7324284	.7101169
.29	.4764413	.4339495	.3960196	.79	.7625540	.7392069	.7174021
.30	.4829331	.4406840	.4029116	.80	.7688748	.7460967	.7248112
.31	.4893316	.4473321	.4097267	.81	.7753026	.7531077	.7323560
.32	.4956425	.4539000	.4164705	.82	.7818495	.7602529	.7400497
.33	.5018722	.4603935	.4231485	.83	.7885278	.7675460	.7479073
.34	.5080266	.4668183	.4297656	.84	.7953520	.7750028	.7559460
.35	.5141098	.4731785	.4363254	.85	.8023385	.7826417	.7641864
.36	.5201274	.4794785	.4428332	.86	.8095069	.7904840	.7726508
.37	.5260835	.4857231	.4492924	.87	.8168805	.7985555	.7813678
.38	.5319828	.4919162	.4557069	.88	.8244864	.8068863	.7903700
.39	.5378288	.4980616	.4620807	.89	.8323582	.8155129	.7996970
.40	.5436252	.5041628	.4684167	.90	.8405365	.8244806	.8093985
.41	.5493761	.5102239	.4747186	.91	.8490732	.8338465	.8195361
.42	.5550854	.5162477	.4809893	.92	.8580347	.8436838	.8301899
.43	.5607549	.5222378	.4872322	.93	.8675088	.8540894	.8414656
.44	.5663887	.5281965	.4934502	.94	.8776165	.8651971	.8535082
.45	.5719893	.5341268	.4996456	.95	.8885322	.8771990	.8665273
.46	.5775608	.5400325	.5058219	.96	.9005239	.8903906	.8808448
.47	.5831047	.5459157	.5119815	.97	.9140428	.9052704	.8970028
.48	.5886244	.5517796	.5181271	.98	.9299712	.9228111	.9160601
.49	.5941222	.5576259	.5242614	.99	.9505906	.9455295	.9407555
.50	.5996014	.5634587	.5303868	1.00	1.000000	1.000000	1.000000

The $I_x(p, q)$ Function

$q = 0.5$ $p = .50$ to $.70$

x	$p=$ .50	.60	.70	x	$p=$ .50	.60	.70
B(p,q)	3.141593	2.7745031	2.5057947		3.141593	2.7745031	2.5057947
.01	.0637686	.0379735	.0227433	.51	.5063666	.4527878	.4065981
.02	.0903345	.0576663	.0370236	.52	.5127358	.4595364	.4135910
.03	.1108247	.0736896	.0492781	.53	.5191101	.4663035	.4206160
.04	.1281884	.0877416	.0603989	.54	.5254920	.4730921	.4276764
.05	.1435663	.1005064	.0707605	.55	.5318843	.4799036	.4347748
.06	.1575424	.1123443	.0805656	.56	.5382895	.4867417	.4419128
.07	.1704634	.1234739	.0899398	.57	.5447103	.4936084	.4490941
.08	.1825549	.1340395	.0989678	.58	.5511494	.5005074	.4563214
.09	.1939734	.1441428	.1077098	.59	.5576098	.5074403	.4635974
.10	.2048328	.1538592	.1162111	.60	.5640942	.5144110	.4709253
.11	.2152190	.1632460	.1245067	.61	.5706057	.5214230	.4783086
.12	.2251989	.1723479	.1326241	.62	.5771474	.5284784	.4857498
.13	.2348255	.1812013	.1405862	.63	.5837226	.5355818	.4932539
.14	.2441418	.1898354	.1484110	.64	.5903345	.5427362	.5008236
.15	.2531833	.1982753	.1561146	.65	.5969867	.5499452	.5084635
.16	.2619798	.2065412	.1637101	.66	.6036829	.5572135	.5161774
.17	.2705563	.2146510	.1712089	.67	.6104271	.5645447	.5239704
.18	.2789343	.2226199	.1786208	.68	.6172233	.5719434	.5318468
.19	.2871326	.2304615	.1859550	.69	.6240760	.5794148	.5398124
.20	.2951672	.2381868	.1932187	.70	.6309899	.5869637	.5478718
.21	.3030525	.2458069	.2004193	.71	.6379699	.5945959	.5560322
.22	.3108011	.2533303	.2075625	.72	.6450216	.6023171	.5642993
.23	.3184242	.2607658	.2146543	.73	.6521506	.6101336	.5726806
.24	.3259319	.2681208	.2217001	.74	.6593633	.6180531	.5811834
.25	.3333333	.2754016	.2287039	.75	.6666667	.6260829	.5898160
.26	.3406367	.2826152	.2356706	.76	.6740681	.6342311	.5985883
.27	.3478494	.2897665	.2426040	.77	.6815758	.6425074	.6075099
.28	.3549784	.2968611	.2495079	.78	.6891989	.6509222	.6165923
.29	.3620301	.3039036	.2563858	.79	.6969475	.6594862	.6258479
.30	.3690101	.3108989	.2632412	.80	.7048328	.6682124	.6352907
.31	.3759240	.3178509	.2700768	.81	.7128674	.6771146	.6449362
.32	.3827767	.3247637	.2768956	.82	.7210657	.6862097	.6548027
.33	.3895729	.3316408	.2837010	.83	.7294437	.6955155	.6649100
.34	.3963171	.3384864	.2904952	.84	.7380202	.7050531	.6752819
.35	.4030133	.3453030	.2972809	.85	.7468167	.7148473	.6859450
.36	.4096655	.3520944	.3040609	.86	.7558582	.7249259	.6969310
.37	.4162774	.3588630	.3108370	.87	.7651745	.7353230	.7082773
.38	.4228526	.3656125	.3176118	.88	.7748011	.7460788	.7200286
.39	.4293943	.3723454	.3243879	.89	.7847810	.7572420	.7322387
.40	.4359058	.3790641	.3311675	.90	.7951672	.7688726	.7449745
.41	.4423902	.3857726	.3379527	.91	.8060266	.7810467	.7583203
.42	.4488506	.3924716	.3447459	.92	.8174451	.7938616	.7723840
.43	.4552897	.3991648	.3515493	.93	.8295366	.8074467	.7873092
.44	.4617105	.4058544	.3583647	.94	.8424576	.8219792	.8032923
.45	.4681157	.4125430	.3651944	.95	.8564337	.8377155	.8206178
.46	.4745080	.4192328	.3720406	.96	.8718116	.8550479	.8397213
.47	.4808899	.4259264	.3789058	.97	.8891753	.8746394	.8613367
.48	.4872642	.4326264	.3857918	.98	.9096655	.8977824	.8868971
.49	.4936334	.4393345	.3927010	.99	.9362314	.9278185	.9201049
.50	.5000000	.4460540	.3996360	1.00	1.000000	1.000000	1.000000

The $I_x(p,q)$ Function

$q = 0.5$ $p = .80$ to 1.0

x	$p=$ .80	.90	1.0	x	.80	.90	1.0
B(p, q)	2.2992875	2.1347606	2.000000		2.2992875	2.1347606	2.000000
.01	.0136863	.0082688	.0050126	.51	.3663954	.3311312	.3000000
.02	.0238828	.0154671	.0100505	.52	.3735271	.3383188	.3071797
.03	.0331087	.0223325	.0151142	.53	.3807054	.3455681	.3144345
.04	.0417718	.0290028	.0202041	.54	.3879334	.3528807	.3217670
.05	.0500507	.0355405	.0253206	.55	.3952132	.3602604	.3291796
.06	.0580445	.0419816	.0304640	.56	.4025477	.3677084	.3366750
.07	.0658163	.0483495	.0356349	.57	.4099399	.3752282	.3442561
.08	.0734091	.0546607	.0408337	.58	.4173921	.3828230	.3519259
.09	.0808541	.0609272	.0460608	.59	.4249077	.3904952	.3596876
.10	.0881753	.0671584	.0513167	.60	.4324898	.3982482	.3675445
.11	.0953912	.0733620	.0566019	.61	.4401416	.4060860	.3755002
.12	.1025169	.0795439	.0619168	.62	.4478667	.4140115	.3835586
.13	.1095647	.0857094	.0672621	.63	.4556689	.4220294	.3917237
.14	.1165446	.0918629	.0726382	.64	.4635524	.4301435	.4000000
.15	.1234656	.0980081	.0780456	.65	.4715214	.4383582	.4083920
.16	.1303352	.1041486	.0834849	.66	.4795797	.4466783	.4169048
.17	.1371599	.1102871	.0889566	.67	.4877332	.4551088	.4255437
.18	.1439455	.1164266	.0944615	.68	.4959861	.4636554	.4343146
.19	.1506971	.1225693	.1000000	.69	.5043447	.4723238	.4432236
.20	.1574192	.1287174	.1055728	.70	.5128148	.4811201	.4522774
.21	.1641163	.1348737	.1111806	.71	.5214022	.4900523	.4614835
.22	.1707919	.1410391	.1168239	.72	.5301144	.4991264	.4708497
.23	.1774496	.1472163	.1225036	.73	.5389596	.5083517	.4803848
.24	.1840928	.1534067	.1282202	.74	.5479448	.5177364	.4900980
.25	.1907242	.1596119	.1339746	.75	.5570799	.5272899	.5000000
.26	.1973468	.1658340	.1397675	.76	.5663755	.5370244	.5101021
.27	.2039632	.1720740	.1455996	.77	.5758410	.5469501	.5204168
.28	.2105759	.1783337	.1514719	.78	.5854903	.5570816	.5309584
.29	.2171872	.1846145	.1573850	.79	.5953357	.5674319	.5417424
.30	.2237997	.1909180	.1633400	.80	.6053935	.5780193	.5527864
.31	.2304150	.1972454	.1693376	.81	.6156796	.5888605	.5641101
.32	.2370357	.2035984	.1753789	.82	.6262141	.5999773	.5757359
.33	.2436638	.2099783	.1814647	.83	.6370200	.6113940	.5876894
.34	.2503012	.2163866	.1875962	.84	.6481212	.6231374	.6000000
.35	.2569497	.2228247	.1937742	.85	.6595480	.6352395	.6127017
.36	.2636115	.2292939	.2000000	.86	.6713347	.6477369	.6258343
.37	.2702887	.2357957	.2062746	.87	.6835220	.6606746	.6394449
.38	.2769818	.2423317	.2125992	.88	.6961580	.6741049	.6535898
.39	.2836944	.2489033	.2189750	.89	.7093040	.6880913	.6683375
.40	.2904274	.2555121	.2254033	.90	.7230307	.7027128	.6837722
.41	.2971830	.2621593	.2318854	.91	.7374305	.7180686	.7000000
.42	.3037628	.2688471	.2384227	.92	.7526220	.7342862	.7171573
.43	.3107689	.2755765	.2450166	.93	.7687612	.7515343	.7354249
.44	.3176032	.2823495	.2516685	.94	.7860636	.7700454	.7550510
.45	.3244675	.2891679	.2583802	.95	.8048385	.7901537	.7763932
.46	.3313637	.2960333	.2651531	.96	.8255625	.8123723	.8000000
.47	.3382940	.3029472	.2719890	.97	.8490356	.8375651	.8267949
.48	.3452601	.3099120	.2788597	.98	.8768218	.8674173	.8585786
.49	.3522643	.3169293	.2858572	.99	.9129584	.9062814	.9000000
.50	.3593085	.3240014	.2928932	1.00	1.000000	1.000000	1.000000

The $I_x(p, q)$ Function

$q = 0.5$ $p = 1.20$ to 1.30

x	p= 1.20	1.25	1.30	x	p= 1.20	1.25	1.30
B(p,q)	1.791044		1.7079163		1.791044		1.7079163
.01	.0018574		.0011345	.51	.2477474		.2257117
.02	.0042789		.0028016	.52	.2547680		.2326015
.03	.0069799		.0047596	.53	.2618896		.2396036
.04	.0098854		.0069383	.54	.2691146		.2467207
.05	.0129573		.0093005	.55	.2764461		.2539558
.06	.0161721		.0118229	.56	.2838863		.2613124
.07	.0195142		.0144893	.57	.2914383		.2687922
.08	.0229719		.0172878	.58	.2991059		.2764006
.09	.0265367		.0202093	.59	.3068916		.2841390
.10	.0302018		.0232466	.60	.3147995		.2920125
.11	.0339619		.0263940	.61	.3228343		.3000247
.12	.0378126		.0296468	.62	.3309989		.3081808
.13	.0417505		.0330009	.63	.3392990		.3164848
.14	.0457724		.0364532	.64	.3477383		.3249422
.15	.0498761		.0400009	.65	.3563224		.3335584
.16	.0540595		.0436418	.66	.3650570		.3423386
.17	.0583208		.0473736	.67	.3739483		.3512894
.18	.0626586		.0511949	.68	.3830017		.3604180
.19	.0670718		.0551043	.69	.3922251		.3697312
.20	.0715591		.0591004	.70	.4016259		.3792372
.21	.0761202		.0631824	.71	.4112118		.3889443
.22	.0807540		.0673494	.72	.4209920		.3988621
.23	.0854602		.0716008	.73	.4309763		.4090011
.24	.0902387		.0759362	.74	.4411750		.4193720
.25	.0950887		.0803550	.75	.4515998		.4299871
.26	.1000105		.0848571	.76	.4622641		.4408605
.27	.1050039		.0894423	.77	.4731808		.4520065
.28	.1100691		.0941107	.78	.4843680		.4634424
.29	.1152061		.0988623	.79	.4958407		.4751861
.30	.1204153		.1036973	.80	.5076206		.4872589
.31	.1256968		.1086157	.81	.5197282		.4996834
.32	.1310512		.1136181	.82	.5321901		.5124872
.33	.1364789		.1187049	.83	.5450344		.5257003
.34	.1419806		.1238767	.84	.5582944		.5393571
.35	.1475568		.1291340	.85	.5720084		.5534986
.36	.1532084		.1344775	.86	.5862206		.5681710
.37	.1589358		.1399079	.87	.6009847		.5834306
.38	.1647403		.1454263	.88	.6163640		.5993444
.39	.1706225		.1510334	.89	.6324348		.6159931
.40	.1765838		.1567302	.90	.6492921		.6334761
.41	.1826252		.1625180	.91	.6670551		.6519184
.42	.1887478		.1683981	.92	.6858768		.6714818
.43	.1949528		.1743714	.93	.7059602		.6923794
.44	.2012416		.1804397	.94	.7275835		.7149035
.45	.2076157		.1866042	.95	.7511480		.7394756
.46	.2140768		.1928666	.96	.7772681		.7667414
.47	.2206264		.1992287	.97	.8069774		.7977864
.48	.2272663		.2056922	.98	.8422912		.8347259
.49	.2339981		.2122589	.99	.8884081		.8830156
.50	.2408242		.2189312	1.00	1.0000000		1.0000000

The $I_x(p,q)$ Function

$q=0.5$ $\quad$ $p = 1.35$ to 1.45

x	p = 1.35	1.40	1.45	x	p = 1.35	1.40	1.45
B(p,q)	1.6703740	1.6351522	1.6020257		1.6703740	1.6351522	1.6020257
.01	.0008874	.0006944	.0005436	.51	.2155571	.2059298	.1967939
.02	.0022686	.0018378	.0014895	.52	.2223721	.2126637	.2034436
.03	.0039333	.0032518	.0026896	.53	.2293046	.2195208	.2102207
.04	.0058170	.0048791	.0040942	.54	.2363578	.2265037	.2171289
.05	.0078854	.0066885	.0056757	.55	.2435353	.2336167	.2241715
.06	.0101162	.0086598	.0074161	.56	.2508384	.2408605	.2313510
.07	.0124942	.0107787	.0093026	.57	.2582709	.2482399	.2386709
.08	.0150080	.0130346	.0113255	.58	.2658378	.2557588	.2461355
.09	.0176488	.0154194	.0134773	.59	.2735407	.2634196	.2537482
.10	.0204095	.0179265	.0157521	.60	.2813853	.2712272	.2615131
.11	.0232847	.0205506	.0181450	.61	.2893745	.2791867	.2694353
.12	.0262696	.0232873	.0206520	.62	.2975137	.2873013	.2775186
.13	.0293605	.0261329	.0232696	.63	.3058068	.2955766	.2857687
.14	.0325541	.0290844	.0259952	.64	.3142600	.3040184	.2941917
.15	.0358477	.0321393	.0288262	.65	.3228786	.3126311	.3027925
.16	.0392390	.0352953	.0317608	.66	.3316685	.3214223	.3115772
.17	.0427260	.0385506	.0347971	.67	.3406358	.3303978	.3205535
.18	.0463070	.0419035	.0379337	.68	.3497876	.3395653	.3297280
.19	.0499808	.0453527	.0411695	.69	.3591318	.3489316	.3391093
.20	.0537461	.0488972	.0445033	.70	.3686761	.3585057	.3487052
.21	.0576019	.0525360	.0479344	.71	.3784298	.3682969	.3585258
.22	.0615473	.0562683	.0514621	.72	.3884017	.3783144	.3685805
.23	.0655819	.0600936	.0550858	.73	.3986034	.3885694	.3788804
.24	.0697051	.0640113	.0588053	.74	.4090455	.3990735	.3894376
.25	.0739163	.0680212	.0626202	.75	.4197404	.4098393	.4002657
.26	.0782157	.0721231	.0665303	.76	.4307030	.4208819	.4113792
.27	.0826022	.0763167	.0705357	.77	.4419478	.4322158	.4227939
.28	.0870777	.0806023	.0746365	.78	.4534926	.4438601	.4345284
.29	.0916405	.0849798	.0788327	.79	.4653559	.4558328	.4466020
.30	.0962913	.0894495	.0831247	.80	.4775593	.4681573	.4590377
.31	.1010304	.0940117	.0875128	.81	.4901259	.4808563	.4718592
.32	.1058583	.0986667	.0919974	.82	.5030844	.4939588	.4850969
.33	.1107752	.1034150	.0965791	.83	.5164652	.5074972	.4987826
.34	.1157818	.1082574	.1012586	.84	.5303036	.5215068	.5129538
.35	.1208788	.1131943	.1060364	.85	.5446417	.5360308	.5276538
.36	.1260666	.1182264	.1109135	.86	.5595267	.5511180	.5429326
.37	.1313462	.1233548	.1158905	.87	.5750166	.5668273	.5587512
.38	.1367186	.1285801	.1209688	.88	.5911802	.5832292	.5754806
.39	.1421845	.1339035	.1261490	.89	.6080991	.6004074	.5929075
.40	.1477450	.1393261	.1314325	.90	.6258759	.6184667	.6112385
.41	.1534014	.1448489	.1368206	.91	.6446393	.637538[illegible]	.6306080
.42	.1591549	.1504736	.1423145	.92	.6645540	.6577923	.6511885
.43	.1650068	.1562012	.1479156	.93	.6858379	.6794502	.6732085
.44	.1709585	.1620332	.1536256	.94	.7087910	.7028193	.6969807
.45	.1770114	.1679712	.1594459	.95	.7338446	.7283403	.7229561
.46	.1831674	.1740170	.1653785	.96	.7616591	.7566887	.7518244
.47	.1894282	.1801724	.1714252	.97	.7933456	.7890002	.7847455
.48	.1957954	.1864392	.1775879	.98	.8310679	.8274868	.8239784
.49	.2022713	.1928194	.1838686	.99	.8804064	.8778507	.8753457
.50	.2088577	.1993156	.1902699	1.00	1.000000	1.000000	1.010000

The $I_x(p, q)$ Function

$q = 0.5 \quad P = 1.50$ to 1.70

x	P= 1.50	1.60	1.70	x	P= 1.50	1.60	1.70
B(p, q)	1.5707965	1.51336534	1.46171357		1.5707965	1.51336534	1.46171357
.01	.0007257	.0002614	.0001607	.51	.1881204	.1720482	.1575098
.02	.0012077	.0007948	.0005238	.52	.1946807	.1784197	.1636832
.03	.0022255	.0015254	.0010470	.53	.2013737	.1849339	.1700056
.04	.0034369	.0024247	.0017130	.54	.2082024	.1915915	.1764812
.05	.0048182	.0034762	.0025114	.55	.2151699	.1983976	.1831132
.06	.0063536	.0046686	.0034352	.56	.2222797	.2053550	.1899049
.07	.0080318	.0059939	.0044792	.57	.2295352	.2124681	.1968598
.08	.0098443	.0074459	.0056395	.58	.2369403	.2197405	.2039849
.09	.0117844	.0090197	.0069129	.59	.2444990	.2271770	.2112819
.10	.0138468	.0107115	.0082971	.60	.2522155	.2347810	.2187560
.11	.0160272	.0125179	.0097900	.61	.2600945	.2425588	.2264131
.12	.0183220	.0144365	.0113901	.62	.2681408	.2505137	.2342590
.13	.0207281	.0164651	.0130960	.63	.2763598	.2586527	.2422987
.14	.0232430	.0186017	.0149068	.64	.2847570	.2669817	.2505393
.15	.0258646	.0208450	.0168215	.65	.2933384	.2755065	.2589867
.16	.0285911	.0231936	.0188396	.66	.3021105	.2842346	.2676487
.17	.0314210	.0256466	.0209605	.67	.3110804	.2931726	.2765328
.18	.0343530	.0282032	.0231840	.68	.3202554	.3023285	.2856469
.19	.0373861	.0308625	.0255098	.69	.3296437	.3117115	.2950010
.20	.0405193	.0336242	.0279379	.70	.3392541	.3213303	.3046035
.21	.0437521	.0364879	.0304683	.71	.3490960	.3311948	.3144661
.22	.0470837	.0394533	.0331010	.72	.3591800	.3413163	.3245995
.23	.0505139	.0425203	.0358364	.73	.3695172	.3517061	.3350160
.24	.0540424	.0456889	.0386747	.74	.3801201	.3623778	.3457297
.25	.0576689	.0489591	.0416163	.75	.3910022	.3733449	.3567551
.26	.0613934	.0523313	.0446617	.76	.4021785	.3846241	.3681090
.27	.0652160	.0558055	.0478114	.77	.4136655	.3962313	.3798093
.28	.0691369	.0593823	.0510660	.78	.4254815	.4081877	.3918763
.29	.0731562	.0630620	.0544264	.79	.4376470	.4205124	.4043317
.30	.0772743	.0668452	.0578931	.80	.4501849	.4332315	.4172013
.31	.0814916	.0707325	.0614671	.81	.4631209	.4463699	.4305125
.32	.0858087	.0747245	.0651494	.82	.4764843	.4599597	.4442974
.33	.0902262	.0788222	.0689408	.83	.4903085	.4740353	.4585929
.34	.0947447	.0830263	.0728427	.84	.5046316	.4886364	.4734396
.35	.0993650	.0873379	.0768559	.85	.5194980	.5038095	.4888867
.36	.1040880	.0917580	.0809819	.86	.5349594	.5196079	.5049890
.37	.1089147	.0962873	.0852219	.87	.5510771	.5360966	.5218147
.38	.1138459	.1009276	.0895772	.88	.5679242	.5533510	.5394423
.39	.1188830	.1056799	.0940495	.89	.5855892	.5714641	.5579674
.40	.1240271	.1105455	.0986403	.90	.6041813	.5905489	.5775089
.41	.1292794	.1155259	.1033511	.91	.6238377	.6107485	.5982147
.42	.1346415	.1206231	.1081840	.92	.6447345	.6322467	.6202759
.43	.1401147	.1258382	.1131406	.93	.6671049	.6552858	.6439439
.44	.1457008	.1311731	.1182229	.94	.6912688	.6801987	.6695614
.45	.1514014	.1366298	.1234331	.95	.7176856	.7074636	.6976339
.46	.1572183	.1422103	.1287733	.96	.7470601	.7378134	.7289120
.47	.1631535	.1479168	.1342458	.97	.7805761	.7724779	.7646744
.48	.1692091	.1537511	.1398531	.98	.8205388	.8138530	.8074039
.49	.1753872	.1597160	.1455976	.99	.8728886	.8681089	.8634939
.50	.1816901	.1658139	.1514823	1.00	1.0000000	1.0000000	1.0000000

The $I_x(p, q)$ Function

$q = 0.5 \quad P = 1.80$ to 2.00

x	P = 1.80	1.90	2.00
B(p, q)	1.41494618	1.3723461	1.3333333
.01	.0000989	.0000610	.0000376
.02	.0003457	.0002284	.0001510
.03	.0007195	.0004950	.0003409
.04	.0012116	.0008580	.0006082
.05	.0018166	.0013155	.0009536
.06	.0025307	.0018664	.0013779
.07	.0033513	.0025102	.0018821
.08	.0042765	.0032465	.0024670
.09	.0053046	.0040750	.0031335
.10	.0064347	.0049958	.0038825
.11	.0076658	.0060090	.0047150
.12	.0089973	.0071148	.0056319
.13	.0104287	.0083137	.0066341
.14	.0119599	.0096059	.0077228
.15	.0135906	.0109920	.0088990
.16	.0153208	.0124725	.0101636
.17	.0171506	.0140481	.0115180
.18	.0190802	.0157194	.0129630
.19	.0211098	.0174871	.0145000
.20	.0232399	.0193521	.0161301
.21	.0254708	.0213152	.0178545
.22	.0278031	.0233772	.0196745
.23	.0302373	.0255393	.0215915
.24	.0327742	.0278022	.0236066
.25	.0354142	.0301671	.0257214
.26	.0381585	.0326352	.0279372
.27	.0410077	.0352075	.0302556
.28	.0439627	.0378853	.0326779
.29	.0470246	.0406698	.0352059
.30	.0501944	.0435624	.0378410
.31	.0534732	.0465645	.0405849
.32	.0568622	.0496775	.0434395
.33	.0603626	.0529029	.0464064
.34	.0639758	.0562424	.0494875
.35	.0677031	.0596975	.0526847
.36	.0715462	.0632700	.0560000
.37	.0755063	.0669616	.0594354
.38	.0795853	.0707744	.0629931
.39	.0837747	.0747101	.0665752
.40	.0881067	.0787708	.0704840
.41	.0925527	.0824587	.0744219
.42	.0971250	.0872760	.0784915
.43	.1018255	.0917250	.0826951
.44	.1066565	.0963082	.0870356
.45	.1116202	.1010280	.0915157
.46	.1167193	.1058871	.0961383
.47	.1219559	.1108884	.1009064
.48	.1273328	.1160346	.1058233
.49	.1328531	.1213289	.1108922
.50	.1385196	.1267745	.1161165

x	P = 1.80	1.90	2.00
·	1.41494618	1.3723461	1.3333333
.51	.1443350	.1323756	.1215000
.52	.1503035	.1381338	.1270464
.53	.1564277	.1440543	.1327597
.54	.1627111	.1501398	.1386441
.55	.1691588	.1563963	.1447040
.56	.1757737	.1628265	.1509441
.57	.1825609	.1694359	.1573691
.58	.1895243	.1762289	.1639844
.59	.1966690	.1832110	.1707954
.60	.2039998	.1903864	.1778078
.61	.2115223	.1977630	.1850278
.62	.2192430	.2053454	.1924618
.63	.2271671	.2131401	.2001167
.64	.2353022	.2211552	.2080000
.65	.2436540	.2293973	.2161194
.66	.2522317	.2378753	.2244834
.67	.2610427	.2465960	.2331009
.68	.2700951	.2555705	.2419815
.69	.2793995	.2648070	.2511357
.70	.2889651	.2743174	.2605745
.71	.2988036	.2841128	.2703102
.72	.3089266	.2942055	.2803556
.73	.3193473	.3046099	.2907252
.74	.3300791	.3153396	.3014343
.75	.3411386	.3264115	.3125000
.76	.3525430	.3378441	.3239408
.77	.3643104	.3496556	.3357773
.78	.3764628	.3618702	.3480322
.79	.3890221	.3745097	.3607307
.80	.4020156	.3876027	.3739010
.81	.4154717	.4011788	.3875747
.82	.4294238	.4152731	.4017877
.83	.4439105	.4299245	.4165806
.84	.4589740	.4451785	.4320000
.85	.4746650	[illegible]	.4480999
.86	.4910412	.4777085	.4649430
.87	.5081726	.4951177	.4826034
.88	.5261414	.5133982	.5011694
.89	.5450468	.5326538	.5207477
.90	.5650113	.5530114	.5414697
.91	.5861892	.5746297	.5635000
.92	.6087779	.5977141	.5870496
.93	.6330386	.6225338	.6123974
.94	.6593288	.6494586	.6399250
.95	.6881628	.6790203	.6701800
.96	.7203268	.7120304	.7040000
.97	.7571400	.7498522	.7427905
.98	.8011712	.7951359	.7892822
.99	.8590292	.8547016	.8505000
1.00	1.0000000	1.0000000	1.0000000

The $I_x(p, q)$ Function

$q = 0.5$

Miscellaneous values of p corresponding to Profile Fits used in this paper

x	p = .347261	.7994692	.794086	x	p = .347261	.7994692	.794086
B(p, q)	4.0803160	2.3002594	2.3101818		4.0803160	2.3002594	2.3101818
.01	.1427877	.0137231	.0141019	.51	.6071199	.3665951	.3686270
.02	.1818832	.0239382	.0245072	.52	.6125470	.3737262	.3757523
.03	.2096584	.0331782	.0338926	.53	.6179621	.3809040	.3829243
.04	.2319922	.0418532	.0426879	.54	.6233685	.3881313	.3901449
.05	.2510183	.0501422	.0510803	.55	.6287684	.3954104	.3974165
.06	.2677842	.0581450	.0591742	.56	.6341641	.4027442	.4047421
.07	.2828922	.0659247	.0670354	.57	.6395582	.4101353	.4121235
.08	.2967235	.0735247	.0747091	.58	.6449534	.4175867	.4195657
.09	.3095368	.0809764	.0822279	.59	.6503519	.4251009	.4270695
.10	.3215170	.0883037	.0896167	.60	.6557562	.4326820	.4346387
.11	.3328013	.0955252	.0968951	.61	.6611701	.4403325	.442276[illegible]
.12	.3434938	.1026560	.1040784	.62	.6665949	.4480568	.449988[illegible]
.13	.3536778	.1097087	.1111798	.63	.6720340	.4558575	.4577752
.14	.3634175	.1166930	.1182095	.64	.6774905	.4637395	.4656427
.15	.3727677	.1236182	.1251771	.65	.6829672	.4717069	.4735943
.16	.3817722	.1304916	.1320901	.66	.6884671	.4797636	.4816351
.17	.3904679	.1373200	.1389552	.67	.6939934	.4879155	.4897695
.18	.3988863	.1441089	.1457789	.68	.6995499	.4961667	.4980025
.19	.4070547	.1508637	.1525660	.69	.7051400	.5045234	.5063405
.20	.4149956	.1575889	.1593215	.70	.7107675	.5129913	.5147885
.21	.4227298	.1642887	.1660497	.71	.7164364	.5215768	.5233532
.22	.4302744	.1709670	.1727545	.72	.7221513	.5302869	.5320419
.23	.4376459	.1776272	.1794396	.73	.7279164	.5391296	.5408618
.24	.4448579	.1842726	.1861083	.74	.7337371	.5481128	.5498211
.25	.4519219	.1909061	.1927635	.75	.7396188	.5572455	.5589292
.26	.4588504	.1975307	.1994084	.76	.7455676	.5665383	.5681958
.27	.4656526	.2041490	.2060454	.77	.7515891	.5760014	.5776317
.28	.4723383	.2107636	.2126776	.78	.7576916	.5856477	.5872502
.29	.4789149	.2173766	.2193067	.79	.7638822	.5954907	.5970632
.30	.4853911	.2239904	.2259357	.80	.7701701	.6055452	.6070870
.31	.4917730	.2306071	.2325661	.81	.7765645	.6158283	.6173379
.32	.4980673	.2372290	.2392008	.82	.7830769	.6263596	.6278355
.33	.5042799	.2438582	.2458416	.83	.7897199	.6371616	.6386024
.34	.5104166	.2504966	.2524906	.84	.7965077	.6482593	.6496632
.35	.5164825	.2571461	.2591496	.85	.8034570	.6596825	.6610475
.36	.5224823	.2638087	.2658204	.86	.8105869	.6714650	.6727895
.37	.5284202	.2704863	.2725055	.87	.8179204	.6836480	.6849296
.38	.5343004	.2771806	.2792064	.88	.8254851	.6962802	.6975169
.39	.5401278	.2838936	.2859251	.89	.8333138	.7094208	.7106096
.40	.5459054	.2906271	.2926633	.90	.8414471	.7231429	.7242806
.41	.5516372	.2973831	.2994230	.91	.8499366	.7375373	.7386209
.42	.5573268	.3041633	.3062061	.92	.8588481	.7527231	.7537487
.43	.5629767	.3109696	.3130145	.93	.8682693	.7688563	.7698193
.44	.5685908	.3178040	.3198502	.94	.8783201	.7861518	.7870466
.45	.5741713	.3246684	.3267149	.95	.8891742	.8049197	.8057395
.46	.5797223	.3315645	.3336106	.96	.9010978	.8256348	.8263712
.47	.5852456	.3384948	.3405395	.97	.9145396	.8490936	.8497386
.48	.5907446	.3454606	.3475033	.98	.9303765	.8768733	.8773978
.49	.5962209	.3524644	.3545042	.99	.9508770	.9129950	.9133672
.50	.6016788	.3595085	.3615447	1.00	1.0000000	1.0000000	1.000000

The $I_x(p,q)$ Function

$q = 0.5$

Miscellaneous values of p corresponding to Profile fits used in this paper

x	p= 1.347261	1.799469	1.794086
B(p,q)	1.6723683	1.4151829	1.4175897
.01	.0008994	.0001015	.0001018
.02	.0022949	.0003534	.0003543
.03	.0039745	.0007340	.0007356
.04	.0058734	.0012341	.0012367
.05	.0079569	.0018480	.0018516
.06	.0102029	.0025720	.0025768
.07	.0125959	.0034032	.0034092
.08	.0151246	.0043396	.0043469
.09	.0177800	.0053796	.0053882
.10	.0205553	.0065219	.0065319
.11	.0234448	.0077658	.0077772
.12	.0264439	.0091104	.0091233
.13	.0295488	.0105553	.0105698
.14	.0327561	.0121002	.0121163
.15	.0360632	.0137450	.0137627
.16	.0394678	.0154895	.0155088
.17	.0429678	.0173338	.0173547
.18	.0465618	.0192781	.0193007
.19	.0502482	.0213226	.0213469
.20	.0540258	.0234676	.0234936
.21	.0578938	.0257136	.0257413
.22	.0618511	.0280610	.0280905
.23	.0658974	.0305105	.0305417
.24	.0700320	.0330626	.0330956
.25	.0742545	.0357181	.0357528
.26	.0785648	.0384777	.0385142
.27	.0829626	.0413422	.0413804
.28	.0874481	.0443126	.0443526
.29	.0920212	.0473898	.0474315
.30	.0966820	.0505749	.0506183
.31	.1014309	.0538689	.0539140
.32	.1062682	.0572730	.0573199
.33	.1111944	.0607885	.0608371
.34	.1162101	.0644166	.0644670
.35	.1213158	.0681589	.0682109
.36	.1265121	.0720166	.0,20703
.37	.1318001	.0759913	.0760467
.38	.1371804	.0800848	.0801418
.39	.1426541	.0842985	.0843572
.40	.1482221	.0886344	.0886947
.41	.1538858	.0930944	.0931562
.42	.1596463	.0976804	.0977437
.43	.1655050	.1023944	.1024593
.44	.1714631	.1072387	.1073051
.45	.1775222	.1122154	.1122833
.46	.1836842	.1173272	.1173966
.47	.1899506	.1225765	.1226472
.48	.1963232	.1279657	.1280378
.49	.2028041	.1334980	.1335714
.50	.2093955	.1391760	.1392509
.51	.2160995	.1444021	.1458783
.52	.2229184	.1503714	.1510597
.53	.2298547	.1564965	.1571959
.54	.2369121	.1627814	.1634912
.55	.2440923	.1692297	.1699497
.56	.2513986	.1758463	.1765757
.57	.2588345	.1826336	.1833724
.58	.2664031	.1895980	.1903455
.59	.2741089	.1967434	.1974995
.60	.2819546	.2040750	.2048386
.61	.2899455	.2115980	.2123704
.62	.2980856	.2193196	.2200985
.63	.3063802	.2272443	.2280303
.64	.3148338	.2353803	.2361718
.65	.3234524	.2437328	.2445381
.66	.3322414	.2523106	.2531132
.67	.3412082	.2611220	.2619285
.68	.3503591	.2701749	.2709357
.69	.3597018	.2794799	.2802933
.70	.3692442	.2890458	.2898520
.71	.3789955	.2988842	.2997020
.72	.3889650	.3090073	.3098260
.73	.3991631	.3194278	.3202471
.74	.4096017	.3301599	.3309790
.75	.4202924	.3412192	.3420370
.76	.4312505	.3526232	.3534389
.77	.4424899	.3643901	.3652029
.78	.4540293	.3765423	.3773509
.79	.4658860	.3891014	.3899845
.80	.4780828	.4020941	.4028916
.81	.4906418	.4155500	.4163393
.82	.5035923	.4295012	.4302826
.83	.5169641	.4439865	.4447539
.84	.5307927	.4590490	.4599103
.85	.5451205	.4747386	.4754877
.86	.5599939	.4911138	.4918485
.87	.5754718	.5082437	.5089628
.88	.5916219	.5262103	.5269119
.89	.6085261	.5451139	.5457956
.90	.6262873	.5650762	.5657362
.91	.6450332	.5862517	.5868371
.92	.6649289	.6088380	.6094455
.93	.6861922	.6330959	.6335721
.94	.7091221	.6593826	.6599239
.95	.7341498	.6882126	.6887136
.96	.7619348	.7203716	.7209263
.97	.7935863	.7571795	.7575785
.98	.8312662	.8012036	.8015341
.99	.8805479	.8590524	.8592693
1.00	1.0000000	1.000000	1.000000

B. Fortran Programs for Calculating Tables of the Incomplete Beta-Function

Abramowitz and Stegun[2] give a series expansion for the I_x function which is equivalent to the following:

$$I_x(p,q) = \frac{x^p(1-x)^q}{p\,B(p,q)}\left[1+\sum_{n=1}^{\infty}\frac{B(p+1,n)}{B(p+q,n)}x^n\right] \tag{11}$$

This converges well if $x < .5$.

For $.5 < x < 1$ the symmetry relation:

$$I_x(a,b) = 1 - I_{(1-x)}(b,a) \tag{12}$$

may be used to evaluate Equation 11 within its region of good convergence. The following Fortran programs use equations 1,11, and 12 and a polynomial approximation[2] to the gamma function to tabulate I_x .

The main program "TABLE" calls the other functions, accepts the input parameters, and prints the output. When TABLE is called it requests a logical unit number for the output with "ENTER IPRNT". When this is typed in, the line "ENTER A, B, N, XO, K"appears. Type in, in free field form, the quantities p, q, N, XO, K where p and q are the quantities we have been using (p and .5), N is the number of terms in the expansion (25 for 6 place accuracy), XO is just smaller than the smallest desired value (XO=0 for a complete table column), and K equals the number of tabular entries desired in the column to be printed. When these are typed in, the line "ENTER DELX" appears. Type in the increment between successive values of x and the program will type out the table with

$$x = XO + I*DELX$$

with $I = 1$ to K

Finally, the line "NEW VALUES, YE or NO" appears. Type in NO to exit the program or YE to repeat.

```
FTN4,L
      PROGRAM TABLE
 5005 FORMAT("ENTER IPRNT")
 5000 FORMAT("ENTER A,B,N,XO,K")
 5012 FORMAT(5X,I4,10X,F14.7)
 5010 FORMAT(5X,F9.6,4X,F14.7)
 5020 FORMAT(7X,"X",20X,"I")
 5030 FORMAT("NEW VALUES, YE OR NO")
 5040 FORMAT(A2)
 5045 FORMAT(7X,"A",20X,"B")
 5047 FORMAT(7X,"N",20X,"K")
 5060 FORMAT("ENTER DELX")
      IYES=2HYE
      IITTY=1
      IOTTY=1
      WRITE(IOTTY,5005)
      READ(IITTY,*)IPRNT
  100 CONTINUE
      WRITE(IOTTY,5060)
      READ(IITTY,*)DELX
      WRITE(IOTTY,5000)
      READ(IITTY,*)A,B,N,XO,K
      WRITE(IPRNT,5045)
      WRITE(IPRNT,5010)A,B
      WRITE(IPRNT,5047)
      WRITE(IPRNT,5048)N,K
 5048 FORMAT(5X,I4,10X,I14)
      WRITE(IPRNT,5020)
      DO 200 I=1,K
      X=XO+DELX*I
      OUT=BI(X,A,B,N)
      WRITE(IPRNT,5010)X,OUT
  200 CONTINUE
      WRITE(IOTTY,5030)
      READ(IITTY,5040)IQUER
      IF(IQUER.EQ.IYES)GO TO 100
      END

      FUNCTION BI(X,A,B,N)
      XA=X**A
      XB=(1.0-X)**B
      FCTR=XA*XB/(A*BETA(A,B))
      IF(X.GT.0.50)FCTR=XA*XB/(B*BETA(A,B))
      BIN=1.0
      AP=A+1.0
      IF(X.GT.0.50)AP=B+1.0
      AB=A+B
      XQ=X
      IF(X.GT.0.50)XQ=1.0-X
      DO 10 I=1,N
      XI=XQ**I
      YI=I
      BE1=BETA(AP,YI)
      BE2=BETA(AB,YI)
      BIN=BIN+XI*BE1/BE2
   10 CONTINUE
      BI=BIN*FCTR
      IF(X.GT.0.50)BI=1.0-BI
      RETURN
      END
```

```
      FUNCTION BETA(P,Q)
      PG=GAMMA(P)
      QG=GAMMA(Q)
      PQG=GAMMA(P+Q)
      BETA=PG*QG/PQG
      RETURN
      END

      FUNCTION GAMMA(A)
      DOUBLE PRECISION AK(8),RG,AD,ALD,DJ,AI,AG
      DATA AK/-.577191652D0,.988205891D0,
     1 -.897056937D0,.918206857D0,-.756704078D0,
     2 .482199394D0,-.193527818D0,.035868343D0/
      AD=A
      AG=1.0
      RG=1.0
      IF(AD.GE.1.0D0)GO TO 19
      DO 10 I=1,8
      RG=RG+AD**I*AK(I)
   10 CONTINUE
      RG=RG/AD
      CONTINUE
   15 GAMMA=RG
      RETURN
   19 J=AD
      IF(AD.EQ.1.0D0)J=0
      IF(AD.EQ.2.0D0)J=1
      DJ=J
      ALD=AD-DJ
      DO 25 I=1,8
      RG=RG+ALD**I*AK(I)
   25 CONTINUE
      IF(AD.LE.2.0D0)GO TO 15
      J=J-1
   26 CONTINUE
      AI=J
      AG=AG*(ALD+AI)
      J=J-1
      IF(J.GT.0)GO TO 26
      RG=RG*AG
      GO TO 15
      END
      END$
```

C. Programs for Profile Fitting and Acoustic Ray Plotting in Long Range Deep Ocean Sound Transmission Studies

The following programs written for the Texas Instruments SR56 programmable pocket calculator would be easily adaptable to other programmables using algebraic notation.

Tables C1A and C1B give operating instructions and a listing of a program that finds values of α and β for a Hirsch-Carter model fit to the three points: $C_0, 0; C_1, Z_1$; and $C_2\ Z_2$. The program iterates from a trial value of β to find α and a more accurate value of β. The iteration can be continued to any desired precision. The operating steps 7 and 8 use the final values of α and β to calculate c as a function of z for comparison with the empirical profile.

Tables C2A and C2B give operating instructions and a listing of a program that calculates slope (dc/dz) and sound speed as a function of Z in a Hirsch-Carter type profile with parameters α, β, and C_0.

Tables C3A and C3B give operating instructions and a listing of a program that calculates the depth increment Z that corresponds to a given sound velocity in a Hirsch-Carter type profile with parameters α, β, and C_0.

Tables C4A and C4B give operating instructions and a listing of a program that calculates the depth increment, Z, that corresponds to a given slope dc/dz, in a Hirsch-Carter type profile with parameters α, β, and C_0. The program iterates from a trial value of Z to find a more accurate value. The iteration can be continued to any desired precision.

Tables C5A and C5B give operating instructions and a listing of a program that computes range and travel time of a ray segment or multiple ray segments in a Hirsch-Carter type profile. See Section V of the basic paper for more detail of the equations programmed. The parameters

TABLE C-1A HIRSCH-CARTER PROFILE CURVE FITTING

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL KEY	ENTRIES	IN TABLE C-1B		98 00
			LRN	RST			0
2	PRELOAD REGISTERS	C_0	x^2	STO	1		C_0^2
		C_1	x^2	STO	2		C_1^2
	($\beta_t < \beta$ is a	Z_1	STO	3			Z_1
	test value.	C_2	x^2	STO	7		C_2^2
	Final value will	Z_2	STO	8			Z_2
	replace it)	Δ	STO	9			Δ
		β_t	STO	4			
3	compute β		R/S				β_1
SEE	(β will increment						β_2
NOTE	by Δ until						β_3
	value is just						β_4
	greater than						.....
	ideal. Pauses						β
	show each test.)						β
4	CYCLE FOR NEXT		−	RCL	9	=	$\beta - \Delta$
	SIGNIFICANT		STO	4	RCL	9	
	DIGIT		÷	1	0	=	$\Delta_n = \Delta/10$
			STO	9	RST	R/S	β
	The same as						β_1
	STEP 3						β_2
							β
5	REPEAT 3,4						
	UNTIL β has						
	ENOUGH DIGITS						
6	COMPUTE α		RCL	6	2nd $\sqrt[x]{y}$		
			RCL	4	=		α
7	CALCULATE	Z_a	GTO	7	7	R/S	C_a
	SVP FOR	Z_b	R/S				C_b
	VARIOUS Z_a,	Z_c	R/S				C_c
	Z_b, etc.	Z_d	R/S				C_d
8	REPEAT		RST				
	FITTING						
	PROCESS FOR						
	OTHER BRANCH						
	OF SVP						
	(LOOPS BACK						
	TO STEP 2)						

Note: if β does not change from β_t, either β_t is already larger than β, or trial points 1+2 are in the wrong order in memory. To reverse trial points in memory, key in GTO 57 R/S.

TABLE C-1B HIRSCH·CARTER PROFILE CURVE FITTING

Register Contents

0	1	2	3	4	5	6	7	8	9
	C_0^2	C_1^2	Z_1	β	α^β	α^β_{prev}	C_2^2	Z_2	Δ
Preloaded									

Program

Loc.	Code	Key Entry	Comments	Loc.	Code	Key Entry	Comments
			MAIN LOOP				
00	57	2nd subr		50	01	1	c_0^2
01	03	3	to	51	94	=	$c^2-c_0^2$
02	04	4	calculate	52	30	2nd PROD	α^β
03	33	STO		53	05	5	
04	06	6	α^β from c_1 z_1	54	34	RCL	
05	57	2nd subr	to rotate	55	05	5	α^β
06	05	5	registers	56	58	2nd rtn	
07	07	7	2-7, 3-8	57	34	RCL	This subr
08	57	2nd subr		58	02	2	rotates
09	03	3	α^β from c_2 z_2	59	39	2nd EXC	registers
10	04	4		60	07	7	
11	54	÷		61	33	STO	2 - 7,
12	34	RCL		62	02	2	
13	06	6		63	34	RCL	3 - 8
14	74	−		64	03	3	
15	01	1		65	39	2nd EXC	
16	94	=	$\alpha_2^\beta/\alpha_1^\beta-1$	66	08	8	
17	56	2nd CP	clear test reg.	67	33	STO	
18	47	2nd $x\geq t$	if $\alpha_2^\beta > \alpha_1^\beta$	68	03	3	
19	07	7		69	58	2nd rtn	
20	00	0	Go To 70	70	32	$x \gtrless t$	
21	34	RCL		71	57	2nd subr	
22	04	4	β	72	05	5	restore orig.
23	84	+		73	07	7	pos of registers
24	34	RCL		74	34	RCL	
25	09	9	Δ	75	04	4	β to x reg.
26	94	=	new $\beta = \beta + \Delta$	76	41	R/S	
27	59	2nd Pause	display β	77	45	y^x	2nd LOOP
28	33	STO	STORE new β	78	34	RCL	with Z in x
29	04	4		79	04	4	reg. calculate
30	57	2nd subr	to rotate registers	80	64	×	c
31	05	5	to original	81	34	RCL	
32	07	7	position	82	06	6	
33	42	RST	TO NEXT MAIN LOOP	83	94	=	
34	34	RCL	THIS subr	84	93	+/−	
35	03	3	CALCULATES α^β	85	84	+	
36	45	y^x		86	01	1	
37	34	RCL		87	94	=	
38	04	4		88	54	÷	
39	94	=	z_1^β	89	34	RCL	
40	20	2nd 1/x		90	01	1	
41	33	STO	$1/z_1^\beta$ in 5	91	94	=	
42	05	5		92	48	2nd $\sqrt{x}$	
43	34	RCL		93	20	2nd 1/x	
44	02	2	c_1^2	94	41	R/S	c
45	12	INV		95	22	GTO	2nd LOOP TO
46	30	2nd PROD		96	07	7	calculate next
47	05	5	$1/c_1^2 z_1^\beta$ in 5	97	07	7	c
48	74	−		98			
49	34	RCL		99			

TABLE C-2 A dc/dz & c of HIRSCH-CARTER PROFILE

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL KEY	ENTRIES IN			
			TABLE	C2B			75 00
			LRN	RST			0
2	LOAD REGISTERS	α	R/S				α
		β	R/S				β
		c_o	R/S				c_o
3	CALCULATE dc/dz	Z	R/S				dc/dz
4	CALCULATE c		R/S				c
	REPEAT STEPS 3, 4 WITH NEW Z AS DESIRED						
5	TO CHANGE ALL PARAMETERS CONTINUE AT STEP 2		RST				

TABLE C-2B dc/dz & c OF HIRSCH-CARTER PROFILE

Register Contents

0	1	2	3	4	5	6	7	8	9
	α	β	C_o	Z	αZ				

Program

Loc.	Code	Key Entry	Comments	Loc.	Code	Key Entry	Comments
			LOAD REGISTERS				
00	33	STO	← α	39	34	RCL	
01	01	1		40	05	5	
02	41	R/S	← β	41	45	y^x	
03	33	STO		42	34	RCL	
04	02	2		43	02	2	
05	41	R/S	← C_o	44	53	)	
06	33	STO		45	45	y^x	
07	03	3		46	01	1	
08	41	R/S	← Z	47	92	.	
09	33	STO	MAIN	48	05	5	
10	04	4	LOOP	49	94	=	
11	64	×	THIS SECTION	50	41	R/S	dc/dz →
12	34	RCL	CALCULATES	51	34	RCL	THIS
13	01	1	dc/dz	52	04	4	SECTION
14	94	=		53	64	×	CALCULATE
15	33	STO		54	34	RCL	C
16	05	5		55	01	1	
17	45	y^x		56	94	=	
18	52	(		57	45	y^x	
19	34	RCL		58	34	RCL	
20	02	2		59	02	2	
21	74	−		60	74	−	
22	01	1		61	01	1	
23	53	)		62	94	=	
24	64	×		63	93	+/−	
25	34	RCL		64	20	2nd $1/x$	
26	03	3		65	64	×	
27	64	×		66	34	RCL	
28	34	RCL		67	03	3	
29	02	2		68	43	x^2	
30	64	×		69	94	=	
31	34	RCL		70	48	2nd $\sqrt{x}$	
32	01	1		71	41	R/S	c →
33	54	÷		72	22	GTO	← Z
34	02	2		73	00	0	RETURN TO
35	54	÷		74	09	9	MAIN LOOP
36	52	(		75			
37	01	1		76			
38	74	−		77			

TABLE C-3A Depth for a given sound velocity. Hirsch-Carter Profile

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL ENTRIES IN				
			TABLE	C-3B			32 00
			LRN	RST			0
2	LOAD REGISTERS	α	R/S				α
		β	R/S				β
		C_0	R/S				C_0
3	CALCULATE Z	c	R/S				Z
4	REPEAT STEP 3 AS DESIRED						
5	TO LOAD NEW CONSTANTS AND REPEAT STEPS 2, 3		RST				

TABLE C-3B Depth for a given sound velocity. Hirsch-Carter Profile

Register Contents

0	1	2	3	4	5	6	7	8	9
	α	β	C_0						

Program

Loc.	Code	Key Entry	Comments	Loc.	Code	Key Entry	Comments
			LOAD REGISTERS	17	94	=	
				18	93	+/−	$1 - c_0^2/c^2$
00	33	STO	← α	19	45	y^x	
01	01	1		20	34	RCL	
02	41	R/S	← β	21	02	2	
03	33	STO		22	20	2nd 1/x	$(1 - c_0^2/c^2)^{1/\beta}$
04	02	2		23	94	=	
05	41	R/S	← C_0	24	54	÷	
06	33	STO		25	34	RCL	
07	03	3		26	01	1	
08	41	R/S	← c	27	94	=	
09	20	2nd 1/x	MAIN LOOP	28	41	R/S	Z →
10	64	×		29	22	GTO	← c
11	34	RCL		30	00	0	RETURN TO
12	03	3		31	09	9	MAIN LOOP
13	94	=					
14	43	x^2	c_0^2/c^2				
15	74	−					
16	01	1					

TABLE C-4A Depth for a given slope. Hirsch-Carter Profile.

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL KEY	ENTRIES	in		
			Table	C-4B			70 00
			LRN	RST			0
2	LOAD REGISTERS	α	R/S				α
		β	R/S				β
		C_0	R/S				C_0
3	Put dc/dz in t register	dc/dz	$x \gtrless t$				0
4	Initialize ΔZ, Z_t & START	ΔZ	STO	6			ΔZ
		Z_t	R/S				
5	PROGRAM PAUSES AT EACH NEW TRIAL Z HALTS AT FIRST Z with dc/dz G.E. t						Z_1 Z_2 Z_3 Z
			—	RCL	6	=	
6	if desired restart for more accurate Z		2nd EXC	4			dc/dz *
			RCL	6	÷	1	
			0	=			$\Delta Z/10$
			STO	6	RCL	4	
			R/S				
	RECYCLES TO STEP 5						
	* dc/dz here is trial value from last run						

TABLE C-4B Depth for a given slope. Hirsch-Carter Profile

Register Contents

0	1	2	3	4	5	6	7	8	9
	α	β	C_0	Z	αZ	ΔZ			
				dc/dz					

Program

Loc.	Code	Key Entry	Comments
			LOAD REGISTERS
00	33	STO	$\leftarrow \alpha$
01	01	1	
02	41	R/S	$\leftarrow \beta$
03	33	STO	
04	02	2	
05	41	R/S	$\leftarrow C_0$
06	33	STO	
07	03	3	
08	41	R/S	put dc/dz
			in t register,
			ΔZ in 6
			Initiate
			MAIN LOOP
			with trial z
09	33	STO	$\leftarrow Z_t$
10	04	4	
11	64	x	
12	34	RCL	
13	01	1	
14	94	=	αZ_t
15	33	STO	
16	05	5	
17	45	y^x	
18	52	(	
19	34	RCL	
20	02	2	
21	74	−	
22	01	1	
23	53	)	
24	64	x	$(\alpha Z)^{\beta-1}$
25	34	RCL	
26	03	3	
27	64	x	
28	34	RCL	
29	02	2	
30	64	x	
31	34	RCL	
32	01	1	
33	54	÷	
34	02	2	
35	54	÷	$\alpha\beta C_0(\alpha Z)^{\beta-1}/2$

Loc.	Code	Key Entry	Comments
36	52	(	
37	01	1	
38	74	−	
39	34	RCL	
40	05	5	
41	45	y^x	
42	34	RCL	
43	02	2	
44	53	)	$(1-(\alpha Z)^{\beta})$
45	45	y^x	
46	01	1	
47	92	.	
48	05	5	1.5
49	94	=	
			dc/dz
50	12	INV	
51	47	2nd $x \geq t$	test whether
52	06	6	to iterate Z_t
53	00	0	
54	39	2nd EXC	on final
55	04	4	iteration
56	41	R/S	$Z \rightarrow$
			NEW ΔZ
			trial Z
57	22	GTO	REPEAT FOR
58	00	0	MORE ACCURATE Z
59	09	9	
60	34	RCL	
61	04	4	Z_t
62	84	+	
63	34	RCL	
64	06	6	ΔZ
65	94	=	
66	59	2nd Pause	$Z + \Delta Z$
67	22	GTO	To MAIN LOOP
68	00	0	for next
69	09	9	iteration

α , β , c_0 and values B_1 and B_2 of the beta-function (see Section 1 of this supplement are needed for initialization. If the segment is not an integral multiple of the path from reference level to vertex, values of I_1 and I_2 must be entered from a table of the incomplete beta-function (section 1 of this supplement). Otherwise $I_1 = I_2 = 1$. The program recycles for each new value of the reference angle θ_0 , which in a case 1 profile, fitted at the axis, is the axial angle, θ_A .

Tables C6A and C6B give operating instructions and a listing of a program that calculates angles θ_0 and θ_j at points c_0 , O and c_j , z_j of a Hirsch-Carter type profile with parameters α , β . The angles are calculated for given values of the x parameter of the incomplete beta-function to avoid interpolation in the tables when using the previous program (C5A,C5B) in this section.

Tables C7A and C7B give operating instructions and a listing of a program that calculates the characteristic time J (see Milder[3] and section IX of the basic paper) of a sound ray path when the axial sound speed c_0 , the axial angle θ_A , the full cycle range X and the full cycle travel time T are known. The characteristic time is needed to convert axial angles in one profile to those in the next when a ray propagates through a horizontal gradient.

Tables C8A and C8B give operating instructions and a listing of a program that calculates ray angle with the horizontal, and depth increment, of a ray in a Hirsch-Carter type profile. The independent variable is range, which may be specified from an axis crossing or a vertex. The program is initialized with values of α , β , c_0 , and the beta-function, $B\left(\frac{1}{\beta}, \frac{1}{2}\right)$. The ray is designated by its reference angle θ_0 ,

TABLE C-5A Ray plot by Hirsch-Carter profile

NOTE: IF LOC 86 is 2nd rtn Computes Equations 8 - 10
IF. LOC 86 is (computes equations 9 - 11
of the main report

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL KEY	ENTRIES	IN		
			TABLE	C-5B			93 00
			LRN	RST			
2	PRELOAD 2N = 1 single segment 2N = 2 axis to axis or even number for higher order	2N	STO	6			2N
3	LOAD REGISTERS	α	R/S				α
		β	R/S				$2/\beta$
		B_1	R/S				B_1
		B_2	R/S				B_2
		C_0	R/S				C_0
4	RUN MAIN LOOP	θ	R/S				Z_r
		I_1	R/S				R
			R/S				
		I_2	R/S				T
5	REPEAT STEP 4 AS DESIRED WITH ANY θ						
6	TO CHANGE CONSTANTS AND REPEAT STEPS 2, 5		RST				

TABLE C-5B Ray Plot by Hirsch-Carter Profile

Register Contents

0	1	2	3	4	5	6	7	8	9
	θ	$B_1(1-I_1)$ or $B_1 I_1$	α	β	C_0	2×N	2/β	B_1	B_2

Program

Loc.	Code	Key Entry	comments	Loc.	Code	Key Entry	Comments
			LOAD REGISTERS	49	54	÷	
00	33	STO	← α	50	34	RCL	
01	03	3		51	01	1	
02	41	R/S	← β	52	25	tan	
03	33	STO		53	94	=	
04	04	4		54	41	R/S	R →
05	20	2nd 1/x		55	54	÷	
06	64	×		56	34	RCL	
07	02	2		57	05	5	
08	94	=	2/β	58	54	÷	
09	33	STO		59	34	RCL	
10	07	7		60	01	1	
11	41	R/S	← B_1	61	24	cos	
12	33	STO		62	64	×	
13	08	8		63	52	(	
14	41	R/S	← B_2	64	01	1	
15	33	STO		65	74	−	
16	09	9		66	34	RCL	
17	41	R/S	← C_0	67	01	1	
18	33	STO		68	23	sin	
19	05	5		69	43	x^2	
20	41	R/S		70	54	÷	
			MAIN LOOP	71	34	RCL	
21	33	STO	← θ	72	02	2	
22	01	1		73	64	×	
23	23	sin		74	41	R/S	← I_2
24	45	y^x		75	57	2nd subr	for choice
25	34	RCL		76	08	8	of ray segment
26	07	7		77	06	6	
27	54	÷		78	64	×	
28	34	RCL		79	34	RCL	
29	03	3		80	09	9	
30	94	=		81	94	=	
31	64	×		82	41	R/S	T →
32	52	(					← θ
33	41	R/S	Z_v →	83	22	GTO	RETURN TO
			← I_1	84	02	2	BEGIN MAIN
34	57	2nd subr		85	01	1	LOOP AGAIN
35	08	8	for choice of				for segment
36	06	6	ray segment	86	58	2nd rtn	ref. level to
37	64	×					intermediate
38	34	RCL					depth or
39	08	8		86	52	(	for segment
40	53	)	$B_1(\frac{1-I_1}{I_1})$	87	51	CE	intermediate
41	33	STO		88	74	−	level to
42	02	2		89	01	1	vertex
43	64	×		90	53	)	
44	34	RCL		91	93	+/−	(1−I)
45	06	6		9[illegible]	58	2nd rtn	
46	54	÷					
47	34	RCL					
48	04	4					

TABLE C-6A Given x and c_j, find Θ_o and Θ_j. Hirsch-Carter profile

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL KEY	ENTRIES	IN		
			TABLE	C-6B			57 00
			LRN	RST			0
2	LOAD REGISTERS	α	R/S				α
		β	R/S				β
		c_o	R/S				c_o
		z_j	R/S				z_j
		c_j	R/S				c_j
3	RUN MAIN LOOP	x	R/S				Θ_o
			R/S				Θ_j
4	REPEAT STEP 3 WITH NEW X AS DESIRED						
5	FOR NEW CONSTANTS AND REPEAT STEPS 2-4		RST				

TABLE C-6B Given x and c_j, Find θ_0 and θ_j. Hirsch-Carter profile

Register Contents

0	1	2	3	4	5	6	7	8	9
	α	β	C_0	Z_i	θ_0	C_j			

Program

Loc.	Code	Key Entry	Comments	Loc.	Code	Key Entry	Comments
			LOAD REGISTERS	29	52	(	
00	33	STO	← α	30	34	RCL	
01	01	1		31	02	2	
02	41	R/S	← β	32	54	÷	
03	33	STO		33	02	2	
04	02	2		34	53	)	
05	41	R/S	← C_0	35	94	=	$(\alpha z_v)^{\beta/2}$
06	33	STO		36	12	INV	
07	03	3		37	23	sin	
08	41	R/S	← Z_j	38	33	STO	
09	33	STO		39	05	5	
10	04	4		40	41	R/S	θ_0 →
11	41	R/S	← C_j	41	34	RCL	
12	33	STO		42	05	5	
13	06	6		43	24	cos	
14	41	R/S		44	64	×	
			MAIN LOOP	45	34	RCL	
			← X	46	06	6	
15	48	2nd $\sqrt[x]{y}$		47	54	÷	
16	34	RCL		48	34	RCL	
17	02	2		49	03	3	
18	94	=		50	94	=	
19	20	2nd 1/x		51	12	INV	
20	64	×		52	24	cos	
21	34	RCL		53	41	R/S	θ_j →
22	04	4					← x
23	94	=	Z_v	54	22	GTO	LOOP
24	64	×		55	01	1	AGAIN
25	34	RCL		56	05	5	WITH
26	01	1					NEW x
27	94	=					
28	45	y^x					

TABLE C-7A Given full cycle range, X, and Travel Time, T, find characteristic time, J.
See Milder, J. Acoust. Soc. Am., 46, 1259-1263 (1969)

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display
1	KEY IN PROGRAM		2nd CP	LRN			00 00
			ALL	KEY	ENTRIES	IN	
			TABLE	C-7B			18 00
			LRN	RST			0
2	LOAD $1/C_0$	C_0	2nd $1/x$	STO	1		$1/C_0$
3	AXIAL ANGLE	θ_A	R/S				n_m
4	RANGE	X	R/S				$X \cdot n_m$
5	TRAVEL TIME	T	R/S				J
6	REPEAT STEPS 3-5 FOR NEW θ_A AS DESIRED						
7	IF NEW C_0 REPEAT STEPS 2-5						

TABLE C-7B Given Full cycle range, X, & Travel Time, T, Find Char. Time

Register Contents

0	1	2	3	4	5	6	7	8	9
	$1/C_0$ Preloaded								

Program

Loc.	Code	Key Entry	Comments	Loc.	Code	Key Entry	Comments
00	24	cos	← θ_A	10	54	÷	
01	64	×		11	69	2nd π	
02	34	RCL		12	54	÷	
03	01	1		13	02	2	
04	94	=	n_m	14	94	=	
05	64	×		15	93	+/-	
06	41	R/S	← X	16	41	R/S	J →
07	74	−		17	42	RST	
08	41	R/S	← T				
09	94	=					

which for a case 1 profile (see Section IV of the basic paper) fitted at the axis is the axial angle Θ_A. Range may be specified from an axis crossing or a vertex. Tables of the incomplete beta-function are used to convert $I_1 = I_x(1/\rho, \frac{1}{2})$ to x. This program can be used to generate data for a range annotated ray angle diagram as described by Flatte[4] and Cox[5].

TABLE C-8A RANGE ANNOTATED RAY ANGLE DIAGRAM
COMPUTED BY HIRSCH-CARTER MODEL

* If I_1 comes out negative or G.T. 1 range is LARGER than that of reference level to vertex path.

OPERATING INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS				Display	
1	KEY IN PROGRAM		2nd CP	LRN			00 00	
			ALL	KEY	ENTRIES	IN		
			TABLE	C-8B			86 00	
			LRN	RST				
2	LOAD REGISTERS	α	R/S				α	
		β	R/S				β	
		C_0	R/S				C_0	
		B_1	R/S				B_1	
3	SELECT RAY BY REF. ANGLE	θ_0	R/S				Z_v	
4	CONTINUE		R/S					
5	SUPPLY RANGE READ I_1	R	R/S				I_1	*
6	LOOK UP x IN TABLE OF I_x FUNCTION AND ENTER	x	R/S				Z	
7	COMPUTE θ AT RANGE R AND DEPTH Z		R/S				θ	
8	REPEAT STEPS 4-7 AS NECESSARY TO GET DATA FOR PLOT							
9	FOR NEW RAY AND REPEAT STEPS 4-7	θ_0	GTO	1	2	R/S	Z_v	

TABLE C-8B RANGE ANNOTATED RAY ANGLE DIAGRAM

Register Contents.

0	1	2	3	4	5	6	7	8	9
	α	β	C_0	B_1	θ_0	Z_v	Z		

Program

Loc.	Code	Key Entry	Comments
			LOAD REGISTERS
00	33	STO	$\leftarrow \alpha$
01	01	1	
02	41	R/S	$\leftarrow \beta$
03	33	STO	
04	02	2	
05	41	R/S	$\leftarrow C_0$
06	33	STO	
07	03	3	
08	41	R/S	$\leftarrow B_1$
09	33	STO	
10	04	4	
11	41	R/S	SELECT RAY BY REF. ANGLE
12	33	STO	$\leftarrow \theta_0$
13	05	5	
14	23	sin	
15	45	y^x	
16	52	(	
17	02	2	
18	54	÷	
19	34	RCL	
20	02	2	
21	53	)	
22	54	÷	$(\sin\theta_0)^{2/\beta}$
23	34	RCL	
24	01	1	
25	94	=	
26	33	STO	
27	06	6	
28	41	R/S	$Z_v \rightarrow$
			MAIN LOOP STARTS HERE
29	34	RCL	
30	06	6	
31	20	2nd 1/x	
32	54	÷	
33	34	RCL	
34	04	4	
35	64	×	$1/B_1 Z_v$
36	41	R/S	$\leftarrow R$
37	64	×	
38	34	RCL	
39	02	2	
40	64	×	$R\beta/B_1 Z_v$
41	34	RCL	
42	05	5	
43	25	tan	

Loc.	Code	Key Entry	Comments
44	94	=	$R\beta\tan\theta_0/B_1 Z$
45	46	NOP	for R measured from ref. level
46	46	NOP	
47	46	NOP	
48	46	NOP	
			or
45	93	+/−	for R measured from vertex
46	84	+	
47	01	1	
48	94	=	
49	41	R/S	$I_1 \rightarrow$ FIND $\leftarrow \alpha$ IN TABLE
50	40	2nd $\sqrt[x]{y}$	
51	34	RCL	
52	02	2	$x^{1/\beta}$
53	64	×	
54	34	RCL	
55	06	6	
56	94	=	
57	33	STO	
58	07	7	
59	41	R/S	$Z \rightarrow$
60	34	RCL	
61	07	7	
62	64	×	
63	34	RCL	
64	01	1	
65	94	=	αZ
66	45	y^x	
67	34	RCL	
68	02	2	
69	74	−	
70	01	1	
71	94	=	
72	93	+/−	$(1-(\alpha Z)^\beta)$
73	20	2nd 1/x	
74	48	2nd $\sqrt{x}$	
75	64	×	
76	34	RCL	
77	05	5	
78	24	cos	
79	94	=	$\cos\theta$
80	12	INV	
81	24	cos	
82	41	R/S	$\theta \rightarrow$
83	22	GTO	RETURN TO START OF MAIN LOOP
84	02	2	
85	09	9	

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<alpha z> to the beta power

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Formulas for curve fitting and ray computation using compound models made up of several different layers ~~of form~~ $\cancel{c^2 = c_0^2(1 - |\alpha z|^{\beta})^{-1}}$ are presented. Examples of computation by pocket programmable calculator on two Sargasso Sea profiles, one from the center of a cold ring eddy are given. Necessary tables of the incomplete beta-function and calculator programs are included in a supplement.

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